TYPES OF INDETERMINATE FORMS OF LIMITS

- **TYPE** $[0/0]$

(Apply L'Hôpital's Rule.)

EXAMPLE:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

$$\lim_{x\to 0}\frac{\sin x}{x} \overset{[0/0]}{=} \lim_{x\to 0}\frac{\cos x}{1} = \cos(0) = \boxed{1}.$$

- **TYPE** $[\infty/\infty]$

(Apply L'Hôpital's Rule.)

EXAMPLE:

$$\lim_{x\to 0}\frac{\ln(1/x^2)}{\cot(x^2)}$$

Note that $x \to 0$ implies $x^2 \to 0^+$, which implies $1/x^2 \to \infty$, which implies $\ln(1/x^2) \to \infty$. Also, $x^2 \to 0^+$ implies $\tan(x^2) \to 0^+$, which implies $\cot(x^2) = 1/\tan(x^2) \to \infty$. Hence this limit is indeterminate of type $[\infty/\infty]$, so

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(1/x^2)}{\cot(x^2)} &= \lim_{x\to 0}\frac{\ln(x^{-2})}{\cot(x^2)} \\
&= \lim_{x\to 0}\frac{-2\cdot\ln x}{\cot(x^2)} \\
&\overset{[\infty/\infty]}{=} \lim_{x\to 0}\frac{-2/x}{-\csc^2(x^2)\cdot 2x} \\
&= \lim_{x\to 0}\frac{\sin^2(x^2)}{x^2} \\
&\overset{[0/0]}{=} \lim_{x\to 0}\frac{2\,\sin(x^2)\,\cos(x^2)\cdot 2x}{2x} \\
&= \lim_{x\to 0} 2\,\sin(x^2)\,\cos(x^2) \\
&= 2\,\sin(0)\,\cos(0) \\
&= \boxed{0}.
\end{aligned}$$

- **TYPE** $[0\cdot\infty]$

(Use "multiplication = division by reciprocal" to get type $[0/0]$ or $[\infty/\infty]$, then apply L'Hôpital's Rule.)

EXAMPLE:

$$\lim_{x\to 0^+} x\,\ln(1/x)$$

$$\lim_{x\to 0^+} x\,\ln(1/x) = \lim_{x\to 0^+} \frac{\ln(1/x)}{1/x} \overset{[\infty/\infty]}{=} \lim_{x\to 0^+} \frac{\frac{1}{1/x}\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to 0^+} x = \boxed{0}.$$

- **TYPE** $[\infty - \infty]$

(Use algebra to get type $[0/0]$ or $[\infty/\infty]$, then apply L'Hôpital's Rule.)

EXAMPLE:

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$$

$$\begin{aligned}
\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0^+}\frac{\sin x - x}{x\,\sin x} \\
&\overset{[0/0]}{=} \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\,\cos x} \\
&\overset{[0/0]}{=} \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\,\sin x} \\
&= \frac{-\sin(0)}{2\,\cos(0) - 0\cdot\sin(0)} \\
&= \frac{0}{2} = \boxed{0}.
\end{aligned}$$

- **TYPE** $[0^0]$

(Take logarithm to get type $[0\cdot\infty]$, proceed as above, then exponentiate to undo the logarithm.)

EXAMPLE:

$$\lim_{x\to 0^+} x^x$$

Let $L = \lim_{x\to 0^+} x^x$. Then

$$\begin{aligned}
\ln L = \ln\left(\lim_{x\to 0^+} x^x\right) &= \lim_{x\to 0^+}\ln(x^x) = \lim_{x\to 0^+} x\,\ln x \\
&= \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{[\infty/\infty]}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0.
\end{aligned}$$

Therefore $L = e^{\ln L} = e^0 = \boxed{1}$.

- **TYPE** $[\infty^0]$

  (Take logarithm to get type $[0 \cdot \infty]$, proceed as above, then exponentiate to undo the logarithm.)

  EXAMPLE:

$$\lim_{x \to (\pi/2)^+} (\tan x)^{\cos x}$$

  Let $L = \lim_{x \to (\pi/2)^+} (\tan x)^{\cos x}$. Then

$$\begin{aligned}
\ln L &= \ln \left(\lim_{x \to (\pi/2)^+} (\tan x)^{\cos x} \right) \\
&= \lim_{x \to (\pi/2)^+} \ln \left((\tan x)^{\cos x} \right) \\
&= \lim_{x \to (\pi/2)^+} \cos x \, \ln(\tan x) \\
&= \lim_{x \to (\pi/2)^+} \frac{\ln(\tan x)}{\sec x} \\
&\overset{[\infty/\infty]}{=} \lim_{x \to (\pi/2)^+} \frac{(1/\tan x) \cdot \sec^2 x}{\sec x \, \tan x} \\
&= \lim_{x \to (\pi/2)^+} \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} \cdot \frac{\cos x}{\sin x} \cdot \cos x \\
&= \lim_{x \to (\pi/2)^+} \frac{\cos x}{\sin^2 x} \\
&= \frac{\cos(\pi/2)}{(\sin(\pi/2))^2} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

  Therefore $L = e^{\ln L} = e^0 = \boxed{1}$.

- **TYPE** $[1^\infty]$

  (Take logarithm to get type $[0 \cdot \infty]$, proceed as above, then exponentiate to undo the logarithm.)

  EXAMPLE:

$$\lim_{x \to 0^+} (1 + x)^{1/x}$$

Let $L = \lim_{x \to 0^+} (1+x)^{1/x}$. Then

$$\begin{aligned}
\ln L &= \ln\left(\lim_{x \to 0^+} (1+x)^{1/x}\right) \\
&= \lim_{x \to 0^+} \ln\left((1+x)^{1/x}\right) \\
&= \lim_{x \to 0^+} (1/x) \cdot \ln(1+x) \\
&= \lim_{x \to 0^+} \frac{\ln(1+x)}{x} \\
&\overset{[0/0]}{=} \lim_{x \to 0^+} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} \\
&= \lim_{x \to 0^+} \frac{1/(1+x)}{1} \\
&= \frac{1}{1+0} = 1.
\end{aligned}$$

Therefore $L = e^{\ln L} = e^1 = \boxed{e}$.