Quiz 4

Name: ______________________________

Question 1. Please compute $\frac{dy}{dx}$ if

$$y = (\sqrt{x})^x.$$

Question 2. Please find the line tangent to the graph of

$$y = \arcsin\left(\frac{x}{2}\right)$$

when $x = \sqrt{3}$.

(Hint: If you don't remember the derivative of inverse sine, try implicitly differentiating $\sin y = x$ and plugging in $x = \sqrt{3}$, remembering that $-\pi/2 \leq y \leq \pi/2$.)

Quiz 4 Solutions

Solution to Question 1. We use logarithmic differentiation.

$$\begin{aligned}
y &= (\sqrt{x})^x \\
\ln y &= \ln(\sqrt{x})^x \\
\ln y &= x\ \ln(\sqrt{x}) \\
\ln y &= \frac{1}{2}\,x\ \ln(x) \\
\frac{d}{dx}\,[\ln y] &= \frac{d}{dx}\left[\frac{1}{2}\,x\ \ln(x)\right] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{2}\cdot\left(\frac{d}{dx}\,x\right)\cdot\ln(x) + \frac{1}{2}\cdot x\cdot\left(\frac{d}{dx}\,\ln(x)\right) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{2}\,\ln(x) + \frac{1}{2}\cdot x\cdot\left(\frac{1}{x}\right) \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{2}\,\ln(x) + \frac{1}{2} \\
\frac{dy}{dx} &= y\cdot\left(\frac{1}{2}\,\ln(x) + \frac{1}{2}\right) \\
\frac{dy}{dx} &= \boxed{(\sqrt{x})^x\ \left(\frac{1}{2}\,\ln(x) + \frac{1}{2}\right)}
\end{aligned}$$

Solution to Question 2. We the slope and a point on the line. Below we will find the derivative in two different ways, but there is really only one way to find the point. When $x = \sqrt{3}$, we have $y = \arcsin\frac{\sqrt{3}}{2}$, which means we are trying to find the angle y such that $\sin y = \frac{\sqrt{3}}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. (This inequality comes from knowing the **range** of the inverse sine function.) This angle is $y = \frac{\pi}{3}$, so the point of tangency is $(x, y) = \left(\sqrt{3}, \frac{\pi}{3}\right)$.

Now we find a formula for the derivative.

Method 1. (straightforward differentiation, using the Chain Rule)

$$\frac{dy}{dx} = \frac{d}{dx}\arcsin\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1-(x/2)^2}}\cdot\frac{d}{dx}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{1-(x^2/4)}}.$$

$$\left.\frac{dy}{dx}\right|_{x=\sqrt{3}} = \frac{1}{2\sqrt{1-(\sqrt{3})^2/4}} = \frac{1}{2\sqrt{1-(3/4)}} = \frac{1}{2\sqrt{1/4}} = \frac{1}{2(1/2)} = 1.$$

Method 2. (implicit differentiation)

$$\begin{aligned} y &= \arcsin\left(\frac{x}{2}\right) \\ \sin y &= \frac{x}{2} \\ \sin y &= \frac{1}{2}\,x \\ \frac{d}{dx}\,\sin y &= \frac{d}{dx}\,\frac{1}{2}\,x \\ \cos y \cdot \frac{dy}{dx} &= \frac{1}{2} \\ \frac{dy}{dx} &= \frac{1}{2\cos y}. \end{aligned}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(\sqrt{3},\pi/3)} = \left.\left(\frac{1}{2\cos y}\right)\right|_{(x,y)=(\sqrt{3},\pi/3)} = \frac{1}{2\cos(\pi/3)} = \frac{1}{2\cdot(1/2)} = 1.$$

In either case, the tangent line has equation

$$\boxed{y - \frac{\pi}{3} = x - \sqrt{3}}\,.$$