Quiz 5

Name: ______________________________

Question 1.

(a) Use L'Hôpital's Rule to compute $\lim_{x \to 0^+} \ln\left((1+x)^{1/x}\right)$.

(b) Use your answer to part (a) to compute $\lim_{x \to 0^+} (1+x)^{1/x}$.

Question 2.

(a) Sketch the graph of a function $f(x)$ that is **continuous** for $0 \leq x \leq 6$, and satisfies

- $f'(x) > 0$ for $0 < x < 2$, and $f'(x) < 0$ for $2 < x < 6$;
- $f''(x) > 0$ for $0 < x < 4$, and $f''(x) < 0$ for $4 < x < 6$.

(b) What are the **critical points** and **inflection points** of the function you sketched in part (a)?

Quiz 5 Solutions

Solution to Question 1.

(a)

$$\begin{aligned}
\lim_{x\to 0^+} \ln\left((1+x)^{1/x}\right) &= \lim_{x\to 0^+} (1/x)\cdot \ln(1+x) \\
&= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \\
&\overset{[0/0]}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} \\
&= \lim_{x\to 0^+} \frac{1/(1+x)}{1} \\
&= \frac{1}{1+0} \\
&= \boxed{1}.
\end{aligned}$$

(b) Let $L = \lim_{x\to 0^+} (1+x)^{1/x}$. Since the natural logarithm is a continuous function, we have

$$\ln L = \ln\left(\lim_{x\to 0^+} (1+x)^{1/x}\right) = \lim_{x\to 0^+} \ln\left((1+x)^{1/x}\right) = 1.$$

Since $\ln L = 1$, it follows that $L = \mathrm{e}^{\ln L} = \mathrm{e}^1 = \boxed{\mathrm{e}}$.

Solution to Question 2.

(a) The given information implies that $f(x)$ is **increasing** and **concave up** when $0 < x < 2$, **decreasing** and **concave up** when $2 < x < 4$, and **decreasing** and **concave down** when $4 < x < 6$. The graph of an example of such a function is given in Figure 1.

(b) For the graph given in Figure 1, there is a **critical point** at $x = 2$ because $f'(2)$ does not exist, and a critical point at $x = 4$ because $f'(4) = 0$. (Arguably, there is also one at $x = 6$, since it looks like $f(x)$ has a vertical tangent there.) There is an **inflection point** at $x = 4$, because the graph of $f(x)$ both has a tangent line and changes concavity at that point.

I believe that any graph satisfying the properties specified in this problem must have a singular point, and hence a critical point, at $x = 2$, and some kind of critical point at $x = 4$. However, the critical point at $x = 4$ could be because of either a horizontal tangent line or because of no tangent line, depending on the particular graph. This will affect whether or not $x = 4$ is an inflection point, since at an inflection point a graph must have a well-defined tangent line.

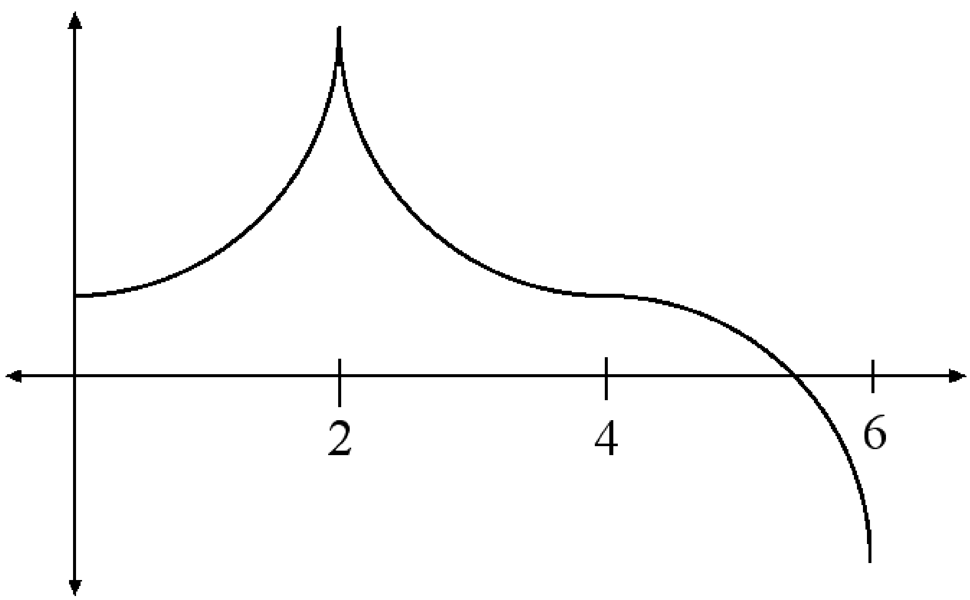

Figure 1: A possible answer to Question 2, part (a).