Quiz 6

Name: ______________________________

Question 1. True or False?
(i.e. Always True or Not Always True)

T **F** The upper sum is always found by using the left endpoint of each subinterval.

T **F** The average value of a nonnegative function $g(x)$ on $[a, b]$ is the area beneath its graph divided by $(b - a)$.

T **F** As we take more, and thinner, rectangles to estimate the area under a curve, the lower sum gets smaller.

T **F** Using the midpoint of each subinterval to obtain our rectangle height always gives an area estimate that is larger than the actual area.

Question 2. Evaluate the following. (Recall that "$\int$" means antiderivative.)

(a) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$

(b) $\displaystyle\lim_{t\to 0} \left(\frac{e^t}{t} - \frac{1}{t}\right)$

(c) $\displaystyle\int (2 - x)^{3/5}\, dx$

Extra Credit. What is the name of the famous mathematician who, as a child, discovered a sneaky way to compute that

$$1 + 2 + \cdots + 99 + 100 = 5,050?$$

Quiz 6 Solutions

Solution to Question 1.

- *The upper sum is always found by using the left endpoint of each subinterval.*

  FALSE. On a particular subinterval, a function *might* attain its maximum at the left endpoint, but it certainly doesn't have to.

- *The average value of a nonnegative function $g(x)$ on $[a, b]$ is the area beneath its graph divided by $(b - a)$.*

  TRUE. This is exactly the definition of the average value of a nonnegative function.

- *As we take more, and thinner, rectangles to estimate the area under a curve, the lower sum gets smaller.*

  FALSE. Generally, as we take more, and thinner, rectangles to estimate the area under a curve, our estimate gets more accurate. But a lower sum is always an underestimate of the true area, and so for lower sums to become more accurate means that they get larger.

- *Using the midpoint of each subinterval to obtain our rectangle height always gives an area estimate that is larger than the actual area.*

  FALSE. This could happen for some functions, but it certainly doesn't happen for all of them.

Solution to Question 2.

(a) **Method 1.**

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{(\sin x)/(\cos x)}{x}\\ &= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{1}{\cos x}\right)\\ &= 1\cdot\frac{1}{\cos(0)}\\ &= 1\cdot 1\\ &= \boxed{1}.\end{aligned}$$

Method 2.

$$\lim_{x\to 0}\frac{\tan x}{x} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\sec^2 x}{1} = \frac{\sec^2(0)}{1} = \frac{1}{\cos^2(0)} = \boxed{1}.$$

(b)

$$\lim_{t\to 0}\left(\frac{e^t}{t}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{e^t-1}{t}\overset{\text{L'H}}{=}\lim_{t\to 0}\frac{e^t}{1}=\frac{e^0}{1}=\boxed{1}.$$

(c) Since $\int u^{3/5}\,du=\frac{5}{8}\,u^{8/5}+C$, a good first guess for this antiderivative is $\frac{5}{8}\,(2-x)^{8/5}+C$. However, note that

$$\frac{d}{dx}\left(\frac{5}{8}\cdot(2-x)^{8/5}\right)=\frac{5}{8}\cdot\frac{8}{5}\cdot(2-x)^{3/5}\cdot(-1)=-(2-x)^{3/5}.$$

Therefore the actual antiderivative is

$$\int(2-x)^{3/5}\,dx=\boxed{-\frac{5}{8}\cdot(2-x)^{8/5}+C}.$$

Solution to Extra Credit. German mathematician Carl Friedrich Gauss, 1777–1855.