Evaluating trigonometric functions

Remark. Throughout this document, remember the **angle measurement convention**, which states that if the measurement of an angle appears without units, then it is assumed to be measured in radians.

Contents

Introduction

If you are in this course, then you should already by fairly familiar with trigonometric functions, and how to evaluate them. This document is only meant to be a *very quick* reminder. A thorough discussion of this material can be found in Section B.3 in the appendix to your textbook.

1   Acute and square angles

You will need to memorize the values of sine and cosine at the following acute angles: $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$. You will also need to know their values at the square angles $0 = 0°$, $\frac{\pi}{2} = 90°$, $\pi = 180°$, $\frac{3\pi}{2} = 270°$, and $2\pi = 360°$.

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π
$\sin\theta$	0	$\dfrac{1}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{2}$	0	-1	0	1

The sine and cosine of the square angles are easy to figure out from the definition of cosine and sine as the x and y coordinates of points on the unit circle. (See Section B.3 of the textbook.) The sine and cosine of the acute angles listed above can be found by studying a 30°-60°-90° triangle and a 45°-45°-90° triangle. (See Figure B.22 in Section B.3 of the textbook.)

Values of the other trigonometric functions at the angles listed above can be found easily, since the other functions are all built from sine and cosine.

Example 1.
Question.

Evaluate $\tan \dfrac{\pi}{3}$ and $\sec \dfrac{\pi}{4}$.

Answer.

Since $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$, we have

$$\tan \frac{\pi}{3} = \frac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Similarly,

$$\sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{1/\sqrt{2}} = \sqrt{2}.$$

2   Larger angles — the geometric method

The first thing to notice is that since sine and cosine repeat their values every 2π radians, if you are asked to evaluate one of these functions at an angle which is not between 0 and 2π, you can simply add or subtract 2π until the angle *does* lie in that interval. For example,

$$\sin\left(\frac{10\pi}{3}\right) = \sin\left(\frac{10\pi}{3} - 2\pi\right) = \sin\left(\frac{10\pi}{3} - \frac{6\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right).$$

The next thing to notice is that the value of sine or cosine at any angle θ with $0 \le \theta \le 2\pi$ is essentially determined by their values at an a particular acute angle related to θ, called the **reference angle** of θ.

Definition 2.1.
The **reference angle** of a non-square angle θ is the acute angle formed between the x-axis and the ray from the origin making an angle of θ with the positive x-axis ray.

Recall that the xy-plane is traditionally divided into four **quadrants**, hence dividing the non-square angles in one rotation into four types.

Quadrant	angles θ included	reference angle of θ
I	$0 < \theta < \dfrac{\pi}{2}$	θ
II	$\dfrac{\pi}{2} < \theta < \pi$	$\pi - \theta$
III	$\pi < \theta < \dfrac{3\pi}{2}$	$\theta - \pi$
IV	$\dfrac{3\pi}{2} < \theta < 2\pi$	$2\pi - \theta$

For each acute angle θ, there are exactly four non-square angles between 0 and 2π with reference angle θ: θ, $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$. The key observation is that *angles with the same reference angle have the same sine and cosine, up to sign.* This is because the points on the unit circle corresponding to these angles have the same xy-coordinates, except that they may have the wrong sign. See Figure 1.

The x-coordinates of points in Quadrants I and IV are positive, and those of points in Quadrants II and III are negative. Therefore $\cos\theta$ is positive for θ in Quadrants I and IV, and negative for θ in Quadrants II and III. Similarly, $\sin\theta$ is positive for θ in Quadrants I and II, and negative for θ in Quadrants III and IV. Since $\tan\theta = \sin\theta/\cos\theta$, we know that $\tan\theta$ is positive for θ in Quadrants I and III, and negative for θ in Quadrants II and IV. This data can be combined into the labeling shown in Figure 2. Here "A" stands for "All", "S" for "Sine", "T" for "Tangent", and "C" for "Cosine", and the letter in each quadrant tells you which of $\sin\theta$, $\cos\theta$, and $\tan\theta$ are positive for θ in that quadrant. One way to remember this labeling is with the mnemonic "**A**ll **S**tudents **T**ake **C**alculus." (Other mnemonics for this rule include "**A**ll **S**tudents **T**remble and **C**ower", "**A**ll **S**now **T**astes and **C**old", "**A**ll **S**tations **T**o **C**entral", "**A** **S**mart **T**rig and **C**lass", "**A**dd **S**ugar **T**o and **C**offee", and "**A**ll **S**ilver **T**ea and **C**ups".)
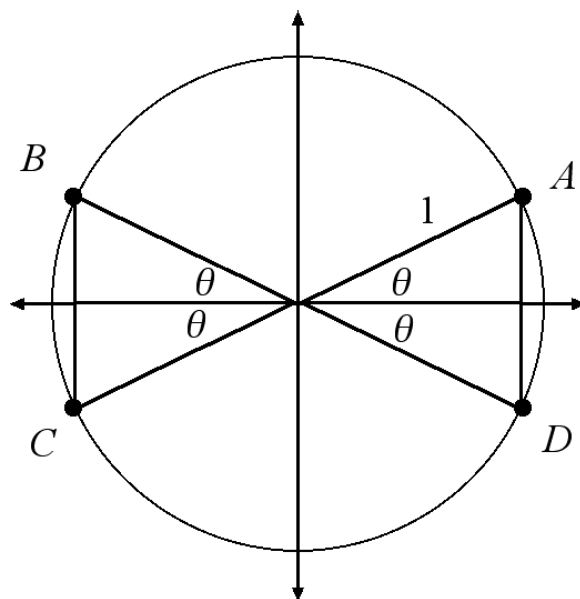
Figure 1: The four points on the unit circle with reference angle θ. Their coordinates are $A = (\cos(\theta), \sin(\theta))$, $B = (\cos(\pi - \theta), \sin(\pi - \theta))$, $C = (\cos(\pi + \theta), \sin(\pi + \theta))$, and $D = (\cos(2\pi - \theta), \sin(2\pi - \theta))$. Notice that these four triangles are congruent to each other.
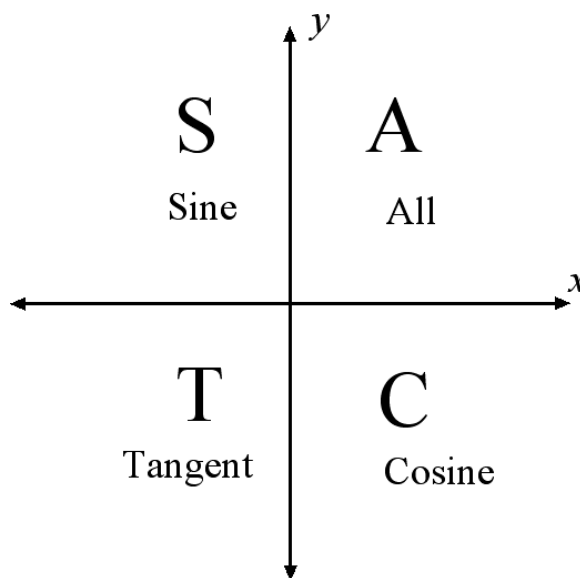

Figure 2: The four Quadrants can be labeled with "**A**ll **S**tudents **T**ake **C**alculus." These labels specify which of sine, cosine, and tangent are positive in that Quadrant.

Example 2.
Question.

Evaluate $\sin \dfrac{5\pi}{3}$ and $\tan \dfrac{5\pi}{4}$.

Answer.

The angle $\dfrac{5\pi}{3}$ is in Quadrant IV, so its reference angle is

$$2\pi - \frac{5\pi}{3} = \frac{6\pi}{3} - \frac{5\pi}{3} = \frac{\pi}{3}.$$

Sine is negative in Quadrant IV, and $\sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$, so $\sin \dfrac{5\pi}{3} = -\dfrac{\sqrt{3}}{2}$.

The angle $\dfrac{5\pi}{4}$ is in Quadrant III, where tangent is positive. Its reference angle is

$$\frac{5\pi}{4} - \pi = \frac{5\pi}{4} - \frac{4\pi}{4} = \frac{\pi}{4}.$$

Since $\tan \dfrac{\pi}{4} = \dfrac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} = \dfrac{1/\sqrt{2}}{1/\sqrt{2}} = 1$, we have $\tan \dfrac{5\pi}{4} = 1$.

3 Larger angles — the formulas method

The second method for evaluating the sine and cosine of larger angles relies on the following useful trigonometric formulas.

$$
\boxed{
\begin{array}{ll}
\cos(-\theta) = \cos\theta & \sin(-\theta) = -\sin\theta \\[2mm]
\cos(\pi - \theta) = -\cos\theta & \sin(\pi - \theta) = \sin\theta
\end{array}
}
$$

Each of these formulas can be derived from geometric properties of the graphs of sine and cosine, or else from those of their definition in terms of the unit circle. For instance, the formula $\sin(-\theta) = -\theta$ expresses the fact that the y-coordinate of the point on the unit circle corresponding to the angle $-\theta$ is the negative of the y-coordinate of the point corresponding to the angle θ. Since $\cos(\pi - \theta) = \cos\left(-(\pi - \theta)\right) = \cos(\theta - \pi)$, the formula

$$\cos(\theta - \pi) = \cos(\pi - \theta) = -\cos\theta$$

can be interpreted as saying that if you move the cosine curve to the right by π, you get exactly an upside-down cosine curve.

As in the first method, notice that any angle greater than 2π or less than 0 can be made into an equivalent angle by adding or subtracting 2π some number of times. At this point, we can use the formulas given above to reduce any given problem to that of evaluating sine or cosine at an acute angle.

Example 3.
Question.

Evaluate $\sin \dfrac{5\pi}{3}$ and $\tan \dfrac{5\pi}{4}$.

Answer.

We compute

$$\sin \frac{5\pi}{3} = \sin \left(\frac{5\pi}{3} - 2\pi \right) = \sin \left(\frac{5\pi}{3} - \frac{6\pi}{3} \right) = \sin \left(-\frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

For the second problem, notice that

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin \theta}{-\cos \theta} = -\tan \theta,$$

and

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta.$$

Therefore

$$\tan \frac{5\pi}{4} = -\tan \left(\pi - \frac{5\pi}{4} \right) = -\tan \left(\frac{4\pi}{4} - \frac{5\pi}{4} \right) = -\tan \left(-\frac{\pi}{4} \right) = \tan \frac{\pi}{4} = 1.$$

(Of course, we could instead have calculated $\sin \dfrac{5\pi}{4}$ and $\cos \dfrac{5\pi}{4}$ individually, and taken their quotient.)

Notice that we obtained the same answers as we did using the other method in Example 2.