WHAT IS A CONNECTION, AND WHAT IS IT GOOD FOR?

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ABSTRACT. In the study of differentiable manifolds, there are several different objects that go by the name of "connection". I will describe some of these objects, and show how they are related to each other. The motivation for many notions of a connection is the search for a sufficiently nice directional derivative, and this will be my starting point as well. The story will by necessity include many supporting characters from differential geometry, all of whom will receive a brief but hopefully sufficient introduction.

I apologize for my ungrammatical title.

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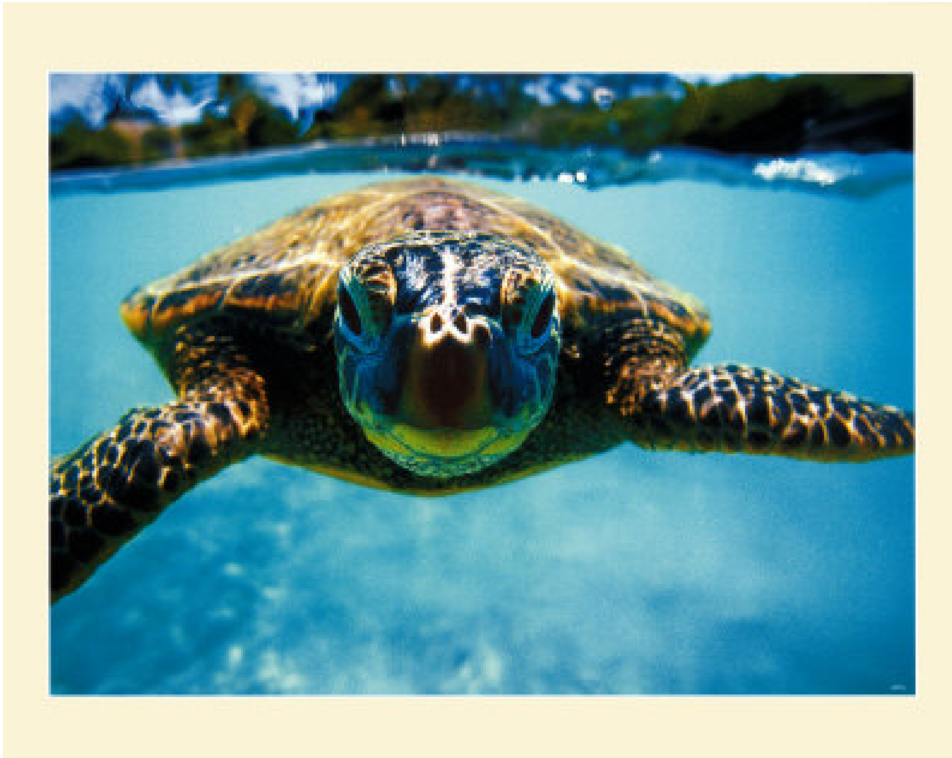

1. Introduction

In the study of differentiable manifolds, there are several different objects that go by the name of "connection", and this has been confusing me for some time now. One solution to this dilemma was to promise myself that I would some day present a talk about connections in the Olivetti Club at Cornell University. That day has come, and this document contains my notes for this talk.

In the interests of brevity, I do not include too many technical details, and instead refer the reader to some lovely references. My main references were [2], [4], and [5]. Some other very interesting references are [3] (which is a truly marvelous book), and [6] (which is also marvelous, although I find it more enigmatic at times). Also, somebody who REALLY knows what he or she is doing has recently been reworking the **Wikipedia** entries having to do with connections and curvature. In preparing this document, I found the entries on *Covariant derivative*, *Connection*, *Koszul connection*, *Ehresmann connection*, and *Connection form* to be very illuminating supplementary material to my textbook reading.

All manifolds in this document, and structures on them and maps between them, are assumed to be smooth.

2. The search for a good directional derivative

Recall from multivariable calculus the definition of the directional derivative. If $g\colon \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, $p \in \mathbb{R}^n$ is a point, and $\vec{v} \in \mathbb{R}^n$ is a tangent vector at p, then the ***directional derivative*** of g at p in the direction $\vec{v}$ is the number $\mathrm{D}_{\vec{v}} g$ defined by

$$\mathrm{D}_{\vec{v}} g := \lim_{t\to 0} \frac{g(p+t\vec{v}) - g(p)}{t}. \tag{1}$$

Remark 2.1. One sometimes thinks of this directional derivative as defining a function, because vectors in $\mathbb{R}^n$ can float, and so $\vec{v}$ can represent a tangent vector to any point in $\mathbb{R}^n$. But this is a peculiarity of vector spaces, and does not hold for arbitrary manifolds. (This is actually a peculiarity of Lie groups, where the single tangent vector $\vec{v}$ can be pushed around by left multiplication to form a vector field. The "multiplication" for a vector space is just addition.) $\Diamond$

Equation (1) doesn't really make sense for an arbitrary manifold, since we can't usually add and subtract vectors from points, so let us use a different definition which actually does work in a more general setting. Let M be a manifold, $g \in C^\infty(M)$ be a function, $p \in M$ be a point, and $\vec{v} \in \mathsf{T}_p M$ be a tangent vector at p. Let $t \mapsto c(t)$ be a smooth curve defined in a neighborhood of 0 such that $c(0) = p$ and $c'(0) = \vec{v}$. Then the ***directional derivative*** of g at p in the direction $\vec{v}$ is the number $\mathrm{D}_{\vec{v}} g$ defined by

$$\mathrm{D}_{\vec{v}} g := \lim_{t\to 0} \frac{g\left(c(t)\right) - g\left(c(0)\right)}{t} = (g \circ c)'(0). \tag{2}$$

One must check that this does not depend on the choice of curve c, but this is not too difficult to do in local coordinates. Just to be sure that Equations (1) and (2) really give the same answer if our manifold is $\mathbb{R}^n$, notice that the curve $c(t) = p + t\vec{v}$ has $c(0) = p$ and $c'(0) = \vec{v}$, and that

$$\frac{g\left(c(t)\right) - g\left(c(0)\right)}{t} = \frac{g(p+t\vec{v}) - g(p)}{t}.$$

If $X \in \mathrm{Vec}(M)$ is a vector field on M and $g \in C^\infty(M)$ is a smooth function, then the ***directional derivative*** of g in the direction X is the *function* $\mathrm{D}_X g$ defined by $\mathrm{D}_X g(p) := \mathrm{D}_{X_p} g$ for $p \in M$. (Here X_p is the tangent vector at x given by the vector field X.) It is not at all obvious that $\mathrm{D}_X g$ has any nice properties at all, but surprisingly, it turns out to be a smooth function (again, use local coordinates). In this way, each vector field $X \in \mathrm{Vec}(M)$ gives rise to a transformation $\mathrm{D}_X \colon C^\infty(M) \to C^\infty(M)$. These transformations have nice properties. Each D_X is linear over $\mathbb{R}$, and obeys a *Leibniz rule* (a.k.a. *product rule*):

$$\mathrm{D}_X(f \cdot g) = f \cdot \mathrm{D}_X g + g \cdot \mathrm{D}_X f \quad \text{for all } f, g \in C^\infty(M). \tag{3}$$

Such transformations of $C^\infty(M)$ are called ***derivations***.

The assignment $X \mapsto D_X$ has another very interesting property. It is clearly linear over $\mathbb{R}$, but because of the point-wise way in which it is defined, it is actually linear over $C^\infty(M)$ also! If $f, g \in C^\infty(M)$, $X \in \mathrm{Vec}(M)$, and $p \in M$ and $g \in C^\infty(M)$, then

$$(\mathrm{D}_{fX}\, g)\,(p) := \mathrm{D}_{(fX)_p}\, g = \mathrm{D}_{(f(p)\cdot X_p)}\, g = f(p) \cdot \mathrm{D}_{X_p}\, g = (f \cdot \mathrm{D}_X g)\,(p),$$

by linearity over $\mathbb{R}$. So $\mathrm{D}_{fX}\, g = f \cdot \mathrm{D}_X g$.

Remark 2.2. Because of the point-wise sort of definition of $\mathrm{D}_X g$, it this linearity of the direction over $C^\infty(M)$ should come as no surprise. The really surprising thing is that the point-wise definition results in a smooth function! $\Diamond$

Having defined a directional derivative for functions, we would like to define one for vector fields.

Definition 2.3. A ***directional derivative*** of vector fields is a map $\nabla \colon \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathrm{Vec}\, M$, $(X, Y) \mapsto \nabla_X Y$, such that ∇ is bilinear over $\mathbb{R}$ and satisfies the *Leibniz rule*,

$$\nabla_X(fY) = \mathrm{D}_X f \cdot Y + f \cdot \nabla_X Y$$

for all $f \in C^\infty(M)$. If ∇ also satisfies $\nabla_{fX} Y = f \cdot \nabla_X Y$ for all $f \in C^\infty(M)$, then ∇ is called ***covariant***, or ***tensorial with respect to direction***.

One defines a directional derivative of differential forms, or more generally of tensor fields, in the same way, with the added requirement that ∇ preserve the degree or type, respectively. $\triangle$

Remark 2.4. The term *tensorial* means exactly linear with respect to scalar multiplication by smooth functions, although the specific definition is usually stated differently. The term *covariant*, on the other hand, usually refers to something quite different. Its use in this context is meant to refer to Riemann's requirement that this derivative be independent of the choice of local coordinates, and hence change *covariantly* with respect to changes in coordinates. Of course, in the modern study of manifolds, most decent transformations or objects are expected to be independent of the choice of local coordinates and hence are all covariant in this sense. Nonetheless, a *covariant derivative* is almost universally understood to mean one that is tensorial in this way. $\Diamond$

Why is tensoriality good? The following proposition answers this question.

Proposition 2.5. *Let X_1, X_2, and Y be vector fields such that X_1 and X_2 agree at a given point $p \in M$. Suppose ∇ is a covariant directional derivative. Then*

$$\nabla_{X_1} Y(p) = \nabla_{X_2} Y(p).$$

Thus the directional derivative at a point depends only on the specific direction at that point.

Proof. Let $e_1, \ldots, e_n \in \mathrm{Vec}(M)$ be a *local frame* for M defined in a neighborhood U of p, i.e. $(e_1)_q, \ldots, (e_n)_q$ form a basis for $\mathsf{T}_q M$ at each $q \in U$. Then over U we can express X_1 and X_2 as linear combinations of $e_1, \ldots, e_n$ over $C^\infty(U)$: $X_1 = \sum_{j=1}^n f_j e_j$ and $X_2 = \sum_{j=1}^n g_j e_j$. By linearity and tensorality of ∇ with respect to direciton, we have

$$\nabla_{X_1} Y = \sum_{j=1}^n f_j \cdot \nabla_{e_j} Y \quad \text{and} \quad \nabla_{X_2} Y = \sum_{j=1}^n g_j \cdot \nabla_{e_j} Y. \tag{4}$$

Because $(X_1)_p = (X_2)_p$, and by the uniqueness of representation of vectors with respect to a basis, we have $f_j(p) = g_j(p)$ for $j = 1, \ldots, n$. Together with (4) above, this implies that $\nabla_{X_1} Y(p) = \nabla_{X_2} Y(p)$. $\mathbb{QED}$

Unfortunately, in general *there is no natural choice of a covariant directional derivative* of vector fields for an arbitrary manifold. There is, however, a natural choice of directional derivative, called the ***Lie derivative***.

Let $X \in \mathrm{Vec}(M)$, and let $\rho_t \colon M \to M$ be the *flow* of X. This means that ρ_0 is the identity map and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \rho_t(p)\right|_{t=0} = X_p$$

for all $p \in M$. It can be shown that $\rho_s \circ \rho_t = \rho_{s+t}$ for $s, t \in \mathbb{R}$ for which the flow is defined. It follows that $\rho_{-t} = (\rho_t)^{-1}$, so that each ρ_t is a diffeomorphism. The ***Lie***

derivative of a vector field $Y \in \mathrm{Vec}(M)$ in the direction X is the vector field $\mathcal{L}_X Y$ defined by

$$(\mathcal{L}_X Y)(p) := \lim_{t \to 0} \frac{Y_p - (\rho_{t*} Y)_p}{t}$$

for all $p \in M$. Here $\rho_{t*} Y$ is the *pushforward* of the vector field Y by the diffeomorphism ρ_t, defined by $(d\phi_t)_{\phi_t^{-1}(p)} Y_{\phi_t^{-1}(p)}$, where $d\phi_t$ is the derivative of ϕ_t.

One can also define the ***Lie derivative*** $\mathcal{L}_X \omega$ of a differential form $\omega \in \Omega^{\cdot}(M)$ on M in the direction X using the flow of X:

$$\left(\mathcal{L}_X \omega\right)(p) := \lim_{t \to 0} \frac{(\rho_t^* \omega)(p) - \omega(p)}{t}$$

for $p \in M$. Here $\rho_t^* \omega$ is the *pullback* of ω by ρ_t, defined by evaluating ω at $\rho_t(p)$ and pushing forward the input tangent vectors by the derivative of ρ_t.

One can show that both of these are directional derivatives, in the sense of Definition 2.3. Unfortunately, neither of them is tensorial with respect to direction. One can show without too much difficulty that $\mathcal{L}_X Y = -\mathcal{L}_Y X$, and so

$$\mathcal{L}_{fX} Y = -\mathcal{L}_Y (fX) = \mathrm{D}_Y f \cdot X + f \cdot \mathcal{L}_Y X = \mathrm{D}_Y f \cdot X - f \cdot \mathcal{L}_X Y.$$

So $\mathcal{L}_X Y$ is tensorial in X if and only if $\mathrm{D}_Y f \cdot X \equiv 0$, which is generally true only if X or Y is the zero vector field. For differential forms, one can use "Cartan's Magic Formula", [2, Proposition 3.10], and the tensoriality of differential forms to deduce

$$\begin{aligned}
\mathcal{L}_{fX} \omega &= (\iota_{fX} \circ \mathrm{d})\omega + (\mathrm{d} \circ \iota_{fX})\omega \\
&= f \cdot (\iota_X \circ \mathrm{d})\omega + \mathrm{d}(f \cdot \iota_X \omega) \\
&= f \cdot (\iota_X \circ \mathrm{d})\omega + \mathrm{d}f \wedge \iota_X \omega + f \cdot (\mathrm{d} \circ \iota_X)\omega \\
&= f \cdot \mathcal{L}_X \omega + \mathrm{d}f \wedge \iota_X \omega.
\end{aligned}$$

Here, $\iota_X \omega$ denotes the *interior product* of X and ω, and which means ω with X plugged into the first slot. Since $\mathrm{d}f \wedge \iota_X \omega \equiv 0$ for all $f \in C^\infty(M)$ in only the most degenerate circumstances, we see that $\mathcal{L}_X \omega$ is almost never tensorial in X.

If our manifold is $\mathbb{R}^n$, then there actually is a natural choice of *covariant* directional derivative. We define $\nabla \colon \mathrm{Vec}(\mathbb{R}^n) \times \mathrm{Vec}(\mathbb{R}^n) \to \mathrm{Vec}(\mathbb{R}^n)$ by

$$\nabla_X Y(p) := \lim_{t \to 0} \frac{Y_{p+tX_p} - Y_p}{t} \tag{5}$$

for $p \in \mathbb{R}^n$. If we write $Y: \mathbb{R}^n \to \mathbb{R}^n$ in terms of its coordinate functions, $Y = (g_1, \dots, g_n)$, then we can write $\nabla_X Y$ in terms of its coordinate functions:

$$\nabla_X Y = (\nabla_X g_1, \dots, \nabla_X g_n) . \tag{6}$$

It is not hard to see that ∇ is tensorial with respect to direction.

Remark 2.6. There are two ways to view the meaning of the validity of this definition. One is that tangent vectors in $\mathbb{R}^n$ are viewed as elements of $\mathbb{R}^n$, which means that a vector field on $\mathbb{R}^n$ gives points in $\mathbb{R}^n$, so that you can compose two vector fields. This is the viewpoint of formula (5).

The other way is to notice that the standard basis for $\mathbb{R}^n$ gives us a *global frame* for $\mathbb{R}^n$, which allows us to express vector fields as ordered tuples of functions, and so boost our covariant directional derivative of functions to a covariant directional derivative of vector fields. This is the viewpoint of (6).

Since the identification of tangent spaces of $\mathbb{R}^n$ with $\mathbb{R}^n$, and the global frame of standard basis vectors, are both structures that are naturally built into the definition of $\mathbb{R}^n$, this covariant directional derivative can be considered to be God-given as well. $\Diamond$

The definition of this natural covariant directional derivative for $\mathbb{R}^n$ highlights the problem with defining one on an arbitrary manifold. For an arbitrary manifold, tangent vectors at different points can't be naturally compared. *There is no natural connection between different tangent spaces.* This is what a ***connection*** will provide, and this is why it is so-named.

Very often, a connection on a manifold is simply defined to be any covariant directional derivative, skipping the intermediate steps. We will approach this goal from the other end.

3. Fiber bundles and Ehresmann connections

Let M and F be topological spaces. A ***fiber bundle*** over M with fiber F is a topological space E and a continuous map $\pi: E \to M$ such that E is locally the product of M and F, in the sense that for every $u \in E$ there is an open neighborhood U of $\pi(p)$ in M and a homeomorphism $\phi: \pi^{-1}(U) \to U \times F$ such that the following diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi", "\cong"'] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$

Here, $\pi_1 \colon U \times F \to U$ is projection onto the first coordinate. From the commuting of the diagram, we see that each ***fiber*** of the bundle, $E_p := \pi^{-1}(p)$, is homeomorphic to F. Thus, one can view E as a collection of "copies" of F parametrized by the points of M. The topology of E comes from the way in which these copies are glued together along M. (This can be made very precise.) The space M is called the ***base space***, and E the ***total space***.

Examples 3.1.

(1) Let $E = M \times F$ and $\pi \colon E \to M$ be projection onto the first coordinate. This is called the ***trivial bundle*** over M with fiber F.

(2) Let $\pi \colon E \to M$ be a covering space. This is a fiber bundle over M with fiber some discrete set F (usually countable).

(3) The cylinder and the Möbius band are both fiber bundles over the circle, with fiber the closed interval $[0, 1] \subset \mathbb{R}$.

$\Diamond$

A ***section*** of the fiber bundle is a (continuous) map $\sigma \colon M \to E$ such that $\pi \circ \sigma = \mathbb{1}_M$. A section picks out an element of each fiber, in a continuous way.

A ***fiber bundle homomorphism*** between two fiber bundles over the same space is a continuous map between them that commutes with the bundle projections. A fiber bundle is called ***trivial*** if it is isomorphic to the trivial bundle with the same fiber. Therefore, every fiber bundle can be called ***locally trivial***.

A very important thing to keep in mind is that even though each fiber is homeomorphic to F by *some* homeomorphism, there is no natural homeomorphism, and so we cannot naturally identify fibers with each other. This is somewhat unfortunate, but it is also where the interesting features of fiber bundles come from. It all of our topological spaces are actually manifolds, then there is a way to compare different fibers. It begins with the following definition.

Definition 3.2. Let $\pi\colon E \to M$ be a smooth fiber bundle with fiber F. For each $u \in E$, let V_u denote the subspace of $\mathsf{T}_u E$ consisting of vectors which are tangent to the fiber $F_{\pi(u)}$. Then $V_u = \mathsf{T}_u(E_{\pi(u)}) = \ker\big(d\pi_u\colon \mathsf{T}_u E \to \mathsf{T}_{\pi(u)}M\big)$. We call V_u the ***vertical subspace*** at u. An ***Ehresmann connection***, or ***fiber bundle connection***, on $\pi\colon E \to M$ is a collection $\Gamma = \{H_u \mid u \in E\}$ of vector subspaces $H_u \subset \mathsf{T}_u E$, called ***horizontal subspaces***, such that

- the assignment $u \mapsto H_u$ depends smoothly on $u \in E$; and
- for each $u \in E$ we have $\mathsf{T}_u E = H_u \oplus V_u$.

$\triangle$

Remark 3.3. An equivalent way to specify a fiber bundle connection is to give a ***vector bundle homomorphism*** $v\colon \mathsf{T}E \to \mathsf{T}E$ such that $v^2 = v$ and $v(\mathsf{T}_u E) = V_u$ for each $u \in E$. Then the horizontal subspaces are given by $H_u := \ker v|_{\mathsf{T}_u E}$. $\diamondsuit$

Example 3.4. Let M and F be manifolds, let $E = M \times F$, and let $\pi\colon E \to M$ be the trivial fiber bundle over M with fiber F. The product structure $E = M \times F$ induces a natural direct sum decomposition $\mathsf{T}E \cong \mathsf{T}M \oplus \mathsf{T}F$ on the tangent bundle of E. Under this identification, the vertical spaces are given by $V_u = \mathsf{T}_u F$ for each $u \in E$. The ***trivial connection*** on this bundle is given by $H_u := \mathsf{T}_u M$ for each $u \in M$.

A connection is said to be ***flat*** at $u \in E$ if there is a local trivialization of the bundle in a neighborhood of u under which the given horizontal subspaces map to the horizontal subspaces of the trivial connection on the corresponding trivial bundle. $\diamondsuit$

Tangent vectors represent directions. Vertical vectors are in the direction of the fibers of the bundle, so if you move in a vertical direction then you remain in the same fiber. The goal of specifying horizontal directions is to tell you exactly how to move from fiber to fiber.

By checking the dimensions of the various vector spaces, we see that $\mathrm{d}\pi_u$ is a linear isomorphism $H_u \to \mathsf{T}_{\pi(u)}M$ for each $u \in E$. Hence, for any $p \in M$ and $u \in E_u$, each vector in $\mathsf{T}_p M$ has a unique horizontal lift to a vector in $\mathsf{T}_u E$. A curve in E is called ***horizontal*** if its velocity vectors are all horizontal.

Given a curve $t \mapsto c(t)$ in M, a ***lift*** of c to E is a curve $t \mapsto \tilde{c}(t)$ in E such that $\pi\left(\tilde{c}(t)\right) = c(t)$ for all t. The curve $\tilde{c}$ is a ***horizontal lift*** if $\tilde{c}$ is a horizontal curve in E.

Theorem 3.5. *Let $\pi\colon E \to M$ be a smooth fiber bundle with fiber F, and let Γ be a connection. If $p \in M$ and $t \mapsto c(t)$ is a smooth curve in M with $c(0) = p$, then for each choice of lift $u \in E_p$ of p, there exists a unique horizontal lift $t \mapsto \tilde{c}(t)$ of c with $\tilde{c}(0) = u$, defined for small t.*

The proof of this theorem involves some results about differential equations, which I will not state here. The words "existence" and "uniqueness" are probably involved. Perhaps also "Picard-Lindelöf Theorem".

The uniqueness of horizontal lifts lets us start with a curve in M through p, and for each u in the fiber over p, travel a short distance to other fibers. We almost have a map between fibers, but the amount of time we can follow the horizontal lift of the curve depends on the initial point chosen in the fiber over p. The basic problem is that an arbitrary fiber bundle, even a smooth fiber bundle, just doesn't have enough structure. In the following sections, we examine special kinds of fiber bundles.

4. A quick word about curvature

Anytime we have a connection, whether it be in terms of horizontal spaces or something else, we have some notion of ***curvature***. This document, and the talk it is based on, are about connections, and curvature is a whole extra (and extremely interesting) conversation. But here is the main idea.

All fiber bundles are *locally trivial*, in that they are locally isomorphic to the trivial bundle over the same base space with the same fiber. But not all connections are locally trivial. A connection on the given bundle may or may not correspond under the local isomorphism to the trivial connection on the trivial bundle. *Curvature is always some sort of mathematical entity that measures how far the given connection is from being trivial.* This mathematical entity usually takes the form of a differential form, or a more general tensor field, or sometimes just a real-valued function. When it is zero, it means the connection, and not just the fiber bundle, is locally trivial. That's why a locally trivial connection is called ***flat***.

Even if the curvature is identically zero, the fiber bundle may not itself be trivial, although its connection will be. Some extra conditions are necessary for this to be true, and what exactly these conditions are depends on the type of connection that is being studies.

5. Principal bundles and principal bundle connections

Let G be a ***Lie group***, which is to say, a manifold with a group structure for which multiplication and inversion are smooth maps. A smooth ***right action*** R of G on a manifold P is a group anti-homomorphism $G \to \mathrm{Diff}(P)$, $g \mapsto R_g$, such that the map $P \times G \to P$ given by $(u, g) \mapsto R_g(u)$ is smooth. (For left actions, $G \to \mathrm{Diff}(P)$ is required to be a homomorphism.)

Definition 5.1. A ***principal*** G***-bundle*** over a manifold M is a fiber bundle $\pi\colon P \to M$ with fiber G together with a smooth right action R of G on P, subject to the following conditions.

- Each R_g preserves fibers, and acts *freely* and *transitively* on each one.
- There is an open cover $\{U_\alpha\}$ of M with local trivializations $\{\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G\}$ such that under ϕ_α, the action of G on $\pi^{-1}(U_\alpha)$ looks like right multiplication; i.e., if $\phi_\alpha(u) = (\pi(u), a)$, then
$$\phi_\alpha\left(R_g(u)\right) = (\pi(u), a \cdot g)\,.$$

$\triangle$

The action R is ***free*** and ***transitive*** on a fiber P_p if for each pair of points $u_1, u_2 \in P_p$, there is a unique $g \in G$ such that $R_g(u_1) = u_2$. (Specifically, ***transitivity*** is the existence and ***freeness*** is the uniqueness.)

Note that $P/G \cong M$.

Example 5.2. (1) For any manifold M and any Lie group G, the trivial fiber bundle $\pi\colon P(= M \times G) \to M$ is a principal G-bundle, with right action R given by $R_g(p, a) = (p, a \cdot g)$ for all $(p, a) \in P = M \times G$ and $g \in G$.

(2) For any Lie group G and closed subgroup H, the quotient projection $\pi\colon G \to G/H$ is a principal H-bundle over G/H (which can be shown to be a manifold), with right action of H on G given by right multiplication.

(3) For any manifold M and point $p \in M$, let Fr_p denote the set of all ***frames*** (ordered bases) of T_pM. The Lie group $\mathrm{GL}(n;\mathbb{R})$ of invertible $(n \times n)$-real matrices acts freely and transitively on each space of frames, by mapping each ordered basis to a new ordered basis. The disjoint union Fr of all the

frame spaces can be given the structure of a fiber bundle over M with fiber $\mathrm{GL}(n;\mathbb{R})$, called the ***frame bundle*** of M, and the action of this Lie group on fibers gives rise to a global action on Fr, and this bundle is a principal $\mathrm{GL}(n;\mathbb{R})$-bundle.

(4) If M is a Riemannian manifold, so that we can measure lengths of tangent vectors and angles between them, then instead of taking the full frame bundle we can restrict our attention to ***orthonormal frames***, and obtain the ***orthonormal frame bundle***. This is a principal $\mathrm{O}(n)$-bundle.

$\diamondsuit$

Because a principal bundle has extra structure, there are extra requirements we can put on a fiber bundle connection.

Definition 5.3. Let $\pi\colon P \to M$ be a G-principal bundle. A ***principal bundle connection*** Γ on this bundle is a fiber bundle connection $\Gamma = \{H_u \mid u \in P\}$ such that the G-action R preserves horizontal subspaces, i.e.

$$H_{R_g(u)} = (\mathrm{d}R_g)_u(H_u) \quad \text{for all } g \in G.$$

$\triangle$

Remark 5.4. An equivalent way to specify a principal bundle connection is to give a $\mathfrak{g}$-valued one-form ω, called a ***connection form***, that satisfies two conditions relating to the G-action on P. (Here, $\mathfrak{g}$ denotes the ***Lie algebra*** of G, which can be identified with the tangent space to G at the identity, $1 \in G$.) Each vector $X \in \mathfrak{g}$ induces a vector field $X^\sharp$on P, called the ***fundamental vector field*** corresponding to X, defined by

$$X^\sharp_u := (\mathrm{d}R^u)_1(X) = \frac{\mathrm{d}}{\mathrm{d}t} R_{\exp tX}(u)\bigg|_{t=0}$$

for $u \in P$, where R^u denotes the function $G \to P$, $g \mapsto R_g(u)$. The form ω is required to satisfy

- $\omega\left(X^\sharp\right) \equiv X$ for all $X \in \mathfrak{g}$; and
- $R_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ for all $g \in G$, where Ad denotes the ***adjoint action*** of G on $\mathfrak{g}$.

The horizontal subspaces are defined by $H_u := \ker \omega_u$ for each $u \in P$. On the other hand, if we begin with the horizontal subspaces H_u, we can obtain a connection form as follows. Notice that each vertical subspace V_u can be described as the space of fundamental vector fields evaluated at u. For a tangent vector $v \in \mathsf{T}_u P$, we define $\omega_u(v)$ to be the unique element $X \in \mathfrak{g}$ such that $X_u^\sharp$ is the vertical component of v. $\Diamond$

Fact 5.5. *The trivial fiber bundle connection on the trivial principal G-bundle $P = M \times G$ over M is a principal bundle connection.*

Here is what all of this extra structure buys us.

Theorem 5.6. *Let $\pi\colon P \to M$ be a principal G-fiber bundle, and let Γ be a principal fiber bundle connection. If $p \in M$ and $c\colon [0,1] \to M$ is a smooth path in M starting at p, then for each choice of lift $u \in P_p$ of p, there exists a unique horizontal lift $\tilde{c}\colon [0,1] \to P$ of c with $\tilde{c}(0) = u$.*

Lifting the path c to a path $\tilde{c}$ starting at the specified point u is nothing unusual. The tricky part is guaranteeing that $\tilde{c}$ is horizontal. The idea of the proof is to adjust $\tilde{c}$ by multiplying via the right action R by a suitable path $g\colon [0,1] \to G$ in G: $t \mapsto R_{g(t)}(\tilde{c}(t))$. The problem of finding this path g reduces to a result about paths in Lie groups, which I will not state, but is not too difficult.

By varying the initial lift point $u \in P_p$, we obtain a map $\tau_t\colon P_p \to P_{c(t)}$ between fibers, called the ***parallel displacement***, or ***parallel transport*** along the curve c.

Proposition 5.7. *Each parallel transportation $\tau_t\colon P_p \to P_{c(t)}$ along a path c starting at $p \in M$ commutes with the action of G on P. Hence τ_t is an isomorphism.*

Proof. Because each R_g preserves both fibers and horizontal subspaces, we see that if $\tilde{c}$ is the horizontal lift of c starting at $u \in P_p$, then $R_g \circ \tilde{c}$ is the horizontal lift of c starting at $R_g(u) \in P_p$. Since this holds all along the curve c, we must have $R_g \circ \tau_t = \tau_t \circ R_g$.

It follows that τ_t is an isomorphism, because G acts freely and transitively on each fiber. $\mathbb{QED}$

Parallel displacement definitely depends on the initial curve c chosen, but it does not depend on the parametrization of this curve. One can check that parallel transportation along the same curve but in the opposing direction is the inverse of the original parallel transportation.

So now we have a true *connection* between different fibers of $\pi\colon P \to M$, although they still depend on the initial path in M that is chosen. (In fact, the curvature of a principal bundle connection can be viewed as measuring *how much* the parallel displacement depends on the choice of curve. This is specifically measured by the ***holonomy groups*** of the connection.)

In the next section, we will learn how to obtain new fiber bundles from principal bundles, and how to share the principal bundle connection with the new bundle.

6. Associated bundles

Let $\pi\colon P \to M$ be a principal G-bundle over M, and let F be a manifold on which G acts smoothly on the *left*, i.e. there exists a group homomorphism $\mathcal{A}\colon G \to \mathrm{Diff}(F)$, $g \mapsto A_g$, such that the map $G \times F \to F$, $(g,\xi) \mapsto A_g(\xi)$ is smooth. We introduce a right action of G on the product $P \times F$ by

$$g \cdot (u,\xi) := (R_g(u), A_{g^{-1}}(\xi))\,.$$

(The inverse is there to change the homomorphism A into an anti-homomorphism.) We denote the quotient space $(P \times F)/G$ by $E := P \times_G F$. Define $\pi_E\colon E \to M$ by $\pi_E\left((u,\xi) \bmod G\right) = \pi(u)$. (This is well-defined because the action of G on P commutes with the projection π.) With a little thought, one can see that each fiber $E_p := \pi_E^{-1}(p)$ is isomorphic to F, and $\pi_E\colon E \to M$ is a fiber bundle over M with fiber F.

As with any quotient, we have a projection $P \times F \to (P \times F)/G =: E$. Given a principal bundle connection Γ_P on P, we can use this projection to obtain a fiber bundle connection on E. Let $w \in E$, and choose $(u,\xi) \in P \times F$ such that $w = (u,\xi) \bmod G$. Write β_ξ for the map $P \to E$ given by $u \mapsto \beta_\xi(u) := (u,\xi) \bmod G$. Define the horizontal subspace $H_w \subset \mathsf{T}_w E$ at w by

$$H_w := (\mathrm{d}\beta_\xi)_u\,(H_u)\,.$$

Observe that this is independent of our choice of (u,ξ), because for all $g \in G$ and $\xi \in F$, we have

$$\beta_\xi(u) = (u,\xi) \bmod G = (R_g(u), A_{g^{-1}}(\xi)) \bmod G = \beta_{A_{g^{-1}}(\xi)}\,(R_g(u))$$

for $u \in P$, so $\beta_{A_{g^{-1}}(\xi)} = \beta_\xi \circ R_{g^{-1}}$. Therefore, if we had chosen $g \cdot (u, \xi) = (R_g(u), A_{g^{-1}}(\xi))$ instead of (u, ξ), then

$$\begin{aligned}
(\mathrm{d}\beta_{A_{g^{-1}}(\xi)})_{R_g(u)} \left(H_{R_g(u)}\right) &= \mathrm{d}\left(\beta_\xi \circ R_{g^{-1}}\right)_{R_g(u)} ((\mathrm{d}R_g)_u(H_u)) \\
&= (\mathrm{d}\beta_\xi)_u \circ (\mathrm{d}R_{g^{-1}})_{R_g(u)} \circ (\mathrm{d}R_g)_u (H_u) \\
&= (\mathrm{d}\beta_\xi)_u (H_u).
\end{aligned}$$

One can show, furthermore, that this collection $\Gamma_E := \{H_w \mid w \in E\}$ satisfy the necessary conditions to be a fiber bundle connection on $\pi_E \colon E \to M$.

Example 6.1. Probably the very most important example of an associated bundle E is where the fiber F is a (finite-dimensional, real) vector space, and $A \colon G \to \mathrm{Diff}(F)$ is a ***linear action***, i.e. $A_g \in \mathrm{GL}(F)$ for all $g \in G$. In this case, E is called a ***vector bundle***. $\diamondsuit$

7. Vector bundles and Koszul connections

Suppose G, $\pi \colon P \to M$, F, $A \colon G \to \mathrm{GL}(F)$, and $\pi_E \colon E \to M$ be as in Example 6.1. One can show that because each A_g is a linear transformation, not only is each fiber $E_p := \pi_E^{-1}(p)$ diffeomorphic to the vector space F, but it inherits a vector space structure all its own. But to define a vector bundle, we don't need to start with a principal bundle.

Definition 7.1. A ***rank*** k ***vector bundle*** over a manifold M is a fiber bundle $\pi_E \colon E \to M$ with fiber $\mathbb{R}^k$ such that each fiber $E_p := \pi_E^{-1}(p)$ has the structure of a vector space, satisfying the following local linear triviality condition.

> For each $w \in E$, there is an open neighborhood U of $\pi_E(w)$ in M and a trivialization $\phi \colon \pi_E^{-1}(U) \to U \times \mathbb{R}^k$ (as in the definition of fiber bundles) such that the map $E_{\pi_E(w)} \to \mathbb{R}^k$ given by the composition
>
> $$w' \mapsto \phi(w') = (\pi_E(w), \xi) \mapsto \xi$$
>
> is a *linear* isomorphism.

$\triangle$

Example 7.2. The absolute most crucial example of a vector bundle is the ***tangent bundle*** $\mathsf{T}M$ of a manifold M, whose fibers are the tangent spaces T_pM, $p \in M$. $\diamondsuit$

Given a vector bundle, we can always work backwards to obtain a principal bundle. If E is a rank k vector bundle, then as in (3) of Example 5.2 above, we can construct the principal $\mathrm{GL}(k;\mathbb{R})$-bundle $\frac{\pi}{\mathrm{Fr}(E)}M$ of frames (ordered bases) of E over M, called the ***frame bundle*** of the vector bundle. As described at the end of the previous section, a principal bundle connection $\Gamma_{\mathrm{Fr}(E)}$ on the frame bundle induces a fiber bundle connection Γ_E on the vector bundle. In this situation, the connection Γ_E will satisfy one extra condition. Each fiber of $\pi_E\colon E \to M$ is a vector space, so we can multiply each element of E by any real number. Therefore, each $\alpha \in \mathbb{R}$ leads to a fiber-preserving diffeomorphism $\bar{\alpha}\colon E \to E$. The horizontal subspaces of E will satisfy

$$(\mathrm{d}\bar{\alpha})_w(H_w) = H_{\bar{\alpha}(w)} = H_{\alpha\cdot w}$$

for all $w \in E$. Any connection on a vector bundle satisfying this property is called a ***vector bundle connection***.

As for a principal bundle, given a smooth path $c\colon [0,1] \to M$ starting at p and choice of lift $w \in E_p$ of p, there exists a unique horizontal lift $\tilde{f}\colon [0,1] \to E$ of c with $\tilde{c}(0) = w$. The proof uses the identification $\left(\mathrm{Fr}(E) \times \mathbb{R}^k\right)/\mathrm{GL}(n;\mathbb{R}) \cong E$, and is apparently not very different from the proof for principal bundles.

By varying the initial lift point $w \in E_p$, we obtain a map $\tau_t\colon E_p \to E_{c(t)}$, the ***parallel transport*** along the curve c. This time, parallel transport is not just an isomorphism of fibers, but it is actually a *linear* isomorphism. That is preserves scalar multiplication is fairly obvious. That it preserves vector addition follows from the following so-called *clever observation* ([5]).

Proposition 7.3. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{0}$ and satisfies $f(\alpha v) = \alpha \cdot f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then f is linear.*

Proof. Since f preserves scalar multiplication, we have $f\left(\vec{0}\right) = \vec{0}$. Let $T = \mathrm{d}f_0\colon \mathbb{R}^n \to \mathbb{R}^m$. Then

$$T(v) = \lim_{\alpha\to 0}\frac{f(\alpha v) - f(0)}{\alpha} = \lim_{\alpha\to 0}\frac{f(\alpha v)}{\alpha} = \lim_{\alpha\to 0} f(v) = f(v).$$

$\mathbb{QED}$

A vector bundle connection on E induces a principal bundle connection on its frame bundle $\mathrm{Fr}(E)$, as follows. Choose $u = (u_1, \ldots, u_k) \in \mathrm{Fr}(E)$, and let $p = \pi_{\mathrm{Fr}(E)}(u)$. Let $c\colon [0,1] \to M$ be a smooth path starting at p and let $\tau_t\colon E_p \to E_{c(t)}$ denote parallel transport along c until time t. Recalling that an element of $\mathrm{Fr}(E)$ can be specified by giving the k vectors, in order, that make up the ordered basis, we define a curve $c^*\colon [0,1] \to \mathrm{Fr}(E)$ by letting the ith vector in $c^*(t)$ be given by

$$(c^*(t))_i := \tau_t(u_i)$$

for $i = 1, \ldots, k$. We define the horizontal subspace to be the space of all vector $(c^*)'(0)$ for different choices of c. One can check that this defines a principal bundle connection, and furthermore, that if we consider E as an associated bundle to the principal bundle $\mathrm{Fr}(E)$, then this principal bundle connection induces the given vector bundle connection.

Finally, we have a linear way to identify fibers of a vector bundle. Now we can differentiate things, and the main thing we would like to differentiate is the sections of the vector bundle. Recall that a section is a map $\sigma\colon M \to E$ such that $\sigma(p) \in E_p$ for all $p \in M$. In general, a fiber bundle may have no globally-defined sections, although local triviality means one can always define a section locally. But a vector bundle always has at least one section, the ***zero section***.

Definition 7.4. Fix a connection on the vector bundle $\pi_E\colon E \to M$. Let $\sigma\colon M \to E$ be a section, and $X \in \mathrm{Vec}(M)$ be a vector field. The ***covariant derivative*** of σ in the direction X is the section $\nabla_X \sigma$ of E defined as follows. Let $p \in M$, and choose any smooth curve $t \mapsto c(t)$ such that $c(0) = p$ and $c'(0) = X_p$, and let τ_t denote parallel transportation along c. Put

$$(\nabla_X \sigma)(p) := \lim_{t \to 0} \frac{\tau_c^{-1}(\sigma(c(t))) - \sigma(p)}{t}.$$

$\triangle$

One has to show that this definition is independent of the choice of curve c. I think this makes sense, because we only need to use parallel transportation τ_t for small t, and so we only need to lift the curve c for very small t, which means that everything really only depends on the initial vector $c'(0)$.

The independence on the choice of c is really what proves that $\nabla_X \sigma$ is tensorial with respect to direction. The Leibniz rule is a relatively simple calculation. Thus, each vector bundle connection gives us a covariant directional derivative of sections, ∇. Such a derivative is called a ***Koszul connection***. Denoting the space of smooth sections of E by $C^\infty(M,E)$, we see that ∇ is a map $\mathrm{Vec}(E) \times C^\infty(M,E) \to C^\infty(M,E)$. Sometimes the inputs are shuffled over to the outputs, and a Koszul connection is defined to be a map

$$C^\infty(M,E) \to \Omega^1(M) \otimes C^\infty(M,E)$$

from smooth sections of E to differential one-forms on M with values in E. This is the path taken in [1].

A section $\sigma\colon M \to E$ is called ***parallel along*** X if $\nabla_X \sigma \equiv 0$, and σ is called ***parallel*** if $\nabla_X \sigma \equiv 0$ for all $X \in \mathrm{Vec}(M)$.

Proposition 7.5. *Let σ be a section of $\pi_E\colon E \to M$. The following are equivalent.*

(1) *σ is parallel.*
(2) *For any curve $t \mapsto c(t)$ on M with parallel transportation τ_t, $\tau_t(\sigma(c(0))) = \sigma(c(t))$ for all t.*
(3) *For all $p \in M$, the image of $\mathsf{T}_p M$ under $(\mathrm{d}\sigma)_p$ is horizontal.*

Equivalence (3) of Proposition 7.5 tells us how we can recover the horizontal subspaces, and hence the vector bundle connection, from the corresponding Koszul connection. The horizontal subspace will be the image of the tangent space in the base manifold under the derivative of all the parallel sections of the vector bundle.

We can also skip straight from a Koszul connection to the principal bundle connection on $\mathrm{Fr}(E)$. This is a little messy, although not difficult, so I will simply point the reader to page 337, Proposition 21 of [5].

I will end this section with a note. In general, there are many ways to define a connection. As is familiar from Riemannian geometry, if the manifold or vector bundle has extra structure, such as a Riemannian metric or a vector bundle metric or a complex structure or whatever, we can pin down which connection we should choose.

8. The tangent bundle

The tangent bundle $\pi\colon \mathsf{T}M \to M$ of a manifold M is the standard example of a vector bundle, and applying all of this vector bundle connection and Koszul connection stuff to it is very fruitful.

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