Representations of Parabolic Groups

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Abstract

Some results on representations of parabolic groups. This is only a reference file.

1 Irreducible representations of Parabolic Groups

Theorem 1 Let \( P \) be a parabolic subgroup, and let \( P = MN \) be its Langlands decomposition. If \((\pi, H)\) is an irreducible unitary representation of \( P \), then

\[
H \cong \text{Ind}^P_{MN} \tau \chi
\]

with \( \tau \in \hat{M}, \chi \in \hat{N} \).

Proof. As an \( N \)-module, we have that

\[
H \cong \int_{\hat{N}} E_\chi d\nu(\chi),
\]

where \( E_\chi \cong L_\chi \otimes V_\chi, V_\chi \in \hat{N}, \) and \( L_\chi \) is a multiplicity space. This means that there exists a vector bundle

\[
\begin{array}{c}
E \\
\downarrow \\
\hat{N}
\end{array}
\]

and a measure \( \nu \) on \( \hat{N} \), such that

\[
H \cong L^2(\hat{N}, E, \nu) := \{ s : \hat{N} \to E | s(\chi) \in E_\chi, \int_{\hat{N}} ||s(\chi)||^2 d\nu(\chi) < \infty \}
\]

under the action

\[
(\pi(n) \cdot s)(\chi) = \chi(n) s(\chi).
\]

Under this isomorphism we can extend this action of \( N \) on \( L^2(\hat{N}, E, \nu) \) to an action of \( P \) on the same space.

Let \( m \in M \), and define

\[
\begin{array}{c}
E^m_\chi \\
\downarrow \\
\hat{N}
\end{array}
\]

to be the vector bundle such that \( E^m_\chi = E_{m, \chi} \). Define a measure \( \nu_m \) on \( \hat{N} \) by

\[
\nu_m(X) = \nu(m \cdot X) \quad \text{for } X \subset \hat{N} \text{ a measurable set}
\]
and define

\[ \tau(m) : L^2(\hat{N}, E, \nu) \rightarrow L^2(\hat{N}, E^m, \nu_m) \]

by

\[ (\tau(m)s)(\chi) = (\phi(m)s)(m \cdot \chi). \]

We claim that \( \tau(m) \) is an isometry. Effectively

\[ \|\tau(m)s\|_m^2 = \int_{\hat{N}} \|\tau(m)(\chi)\|^2 d\nu_m(\chi) \]

\[ = \int_{\hat{N}} \|\pi(m)s(m \cdot \chi)\|^2 d\nu(m \cdot \chi) \]

\[ = \int_{\hat{N}} \|\pi(m)(\chi)\|^2 d\nu(\chi) \]

\[ = \|\pi(m)s\|^2 = \|s\|^2. \]

where the last equality comes from the fact that the action of \( P \) is unitary. Now if we define an action of \( N \) on \( L^2(\hat{N}, E^m, \nu_m) \) by

\[ (\pi(m) \cdot s)(\chi) = \chi(n)s(\chi), \]

then \( \tau(m) \) becomes an \( N \)-intertwiner. Effectively,

\[ \tau(m)(\pi(n)s)(\chi) = \pi(m)\pi(n)s(m \cdot \chi) \]

\[ = \pi(mnm^{-1})\pi(m)s(m \cdot \chi) \]

\[ = \chi(mnm^{-1})\pi(m)s(m \cdot \chi) \]

\[ = \chi(n)\tau(m)s(\chi) = (\pi_m(n)\tau(m)s)(\chi). \]

But now since \( N \) is a CCR group the \( N \)-intertwiner

\[ \tau(m) : L^2(\hat{N}, E, \nu) \rightarrow L^2(\hat{N}, E^m, \nu_m) \]

should come from a morphism of vector bundles

\[ \tilde{\tau}(m) : E \rightarrow E^m, \]

that is, \((\tau(m)s)(\chi) = \tilde{\tau}(m)s(\chi), \) and hence

\[ (\tau(m)s)(\chi) = \tilde{\tau}(m)s(\chi) \]

\[ (\pi(m)s)(m \cdot \chi) = \tilde{\tau}(m)s(\chi) \]

which says that

\[ (\pi(m)s)(\chi) = \tilde{\tau}(m)s(m^{-1} \cdot \chi). \]

Now since \( L^2(\hat{N}, E, \nu) \) is irreducible as a representation of \( P \), the support of \( \nu \) should be contained in a unique \( M \)-orbit on \( \hat{N} \), and hence

\[ L^2(\hat{N}, E, \nu) \cong L^2(M/M\chi, E) \cong \text{Ind}_{M\chi}^P E. \]

Using again that \( L^2(\hat{N}, E, \nu) \) is irreducible we conclude that \( E_{\chi} \cong \tau\chi \) with \( \tau \in M\chi, \chi \in \hat{N} \). Putting all of this together we get that

\[ H \cong \text{Ind}_{M\chi}^P \tau\chi \]

as we wanted to show. \( \blacksquare \)
2 Decomposition of $L^2(P, d_r p)$ under the action of $P \times P$

We will now decompose $L^2(P, d_r p)$ under the action of $P \times P$ given by

$$(p_1, p_2) \cdot f = \delta(p_1)^{-1} L_{p_1} R_{p_2} f.$$

As a left $N$-module

$$L^2(P) \cong \text{Ind}_N^P \text{Ind}_1^N 1 \cong \text{Ind}_N^P (L^2(N))$$

$$\cong \int_N HS(V_\chi) \, d\mu(\chi)$$

$$\cong \int_N \text{Ind}_N^P HS(V_\chi) \, d\mu(\chi) \cong L^2(\hat{N}, E, \mu),$$

with $E_\chi = HS(V_\chi)$. The isomorphism is given in the following way: Given $f \in L^2(P)$, define $s_f \in L^2(\hat{N}, E, \nu)$ by

$$s_f(\chi)(p) = \int_N \chi(n)^{-1} f(np) \, dn.$$

Then

$$s_{R_{p_1} f}(\chi)(p) = \int_N \chi(n)^{-1} R_{p_1} f(np) \, dn = \int_N \chi(n)^{-1} f(np_{p_1}) \, dn$$

$$= s_f(\chi)(pp_1) = (R_{p_1} s_f(\chi))(p),$$

and

$$s_{L_{p_1} f}(\chi)(p) = \int_N \chi(n)^{-1} \delta(p_1)^{-1} L_{p_1} f(np) \, dn$$

$$= \int_N \chi(n)^{-1} \delta(p_1)^{-1} f(p_1^{-1} np_{p_1}^{-1} p) \, dn$$

$$= \int_N \chi(p_1 np_{p_1}^{-1})^{-1} f(np_{p_1}^{-1} p) \, dn$$

$$= \int_N (p_{p_1}^{-1} \chi)(n)^{-1} f(np_{p_1}^{-1} p) \, dn$$

$$= \int_N \left( p_{p_1}^{-1} \chi(p_1^{-1} p) \right) = [L_{p_1}, s_f(p_1^{-1} \chi)](p).$$

Therefore

$$L^2(\hat{N}, E, \mu) \cong L^2(\hat{N}/M, E, \tilde{\mu})$$

$$\cong \int_{\hat{N}/M} \text{Ind}_{\hat{N}/M, N \times P}^P \text{Ind}_N^P HS(V_\chi) \, d\tilde{\mu}(\chi)$$

$$\cong \int_{\hat{N}/M} \text{Ind}_{\hat{N}/M, N}^P (\int_{\hat{N}/M} \tau^* \chi^* \otimes \text{Ind}_{\hat{N}/M}^P \tau \chi) \, d\nu(\tau) \, d\tilde{\mu}(\chi)$$

$$\cong \int_{\hat{N}/M} \int_{\hat{N}/M} \text{Ind}_{\hat{N}/M, N}^P \tau^* \chi^* \otimes \text{Ind}_{\hat{N}/M}^P \tau \chi \, d\nu(\tau) \, d\tilde{\mu}(\chi).$$
3 Decomposition of $L^2(G)$ under the action of $P \times G$

We will now consider $L^2(G)$ as a $P \times G$ module. Reasoning as in the $L^2(P)$ case we have an isomorphism

$$L^2(G) = L^2(\hat{N}, E, \mu)$$

with $E_\chi = \text{Ind}_N^P \text{HS}(V_\chi)$ given in the following way: given $f \in L^2(G)$, define $s_f \in L^2(\hat{N}, E, \mu)$ by

$$s_f(\chi)(g) = \int_\text{N} \chi(n)^{-1} f(ng) \, dn.$$ 

Then

$$s_{R_g f}(\chi)(g) = \int_\text{N} \chi(n)^{-1} R_g f(ng) \, dn = \int_\text{N} \chi(n)^{-1} f(n g g_1) \, dn = s_f(\chi)(g g_1) = (R_g s_f)(\chi)(g).$$

and

$$s_{L_p f}(\chi)(g) = \int_\text{N} \chi(n)^{-1} L_p f(ng) \, dn = \int_\text{N} \chi(n)^{-1} f(p^{-1} n p^{-1} g) \, dn$$

$$= \int_\text{N} \chi(p n p^{-1})^{-1} \delta(p) f(n p^{-1} g) \, dn$$

$$= \int_\text{N} (p^{-1}) \chi(n)^{-1} \delta(p) f(n p^{-1} g) \, dn$$

$$= \delta(p) s_f(p^{-1} \chi)(p^{-1} g) = [\delta(p) L_p s_f(p^{-1} \chi)](g).$$

Therefore

$$L^2(\hat{N}, E, \mu) \cong L^2(\hat{N}/M, E, \tilde{\mu})$$

$$\cong \int_{\hat{N}/M} \text{Ind}_{\hat{M}_x, N \times G}^P \text{Ind}_N^P \text{HS}(V_\chi) \, d\tilde{\mu}(\chi)$$

$$\cong \int_{\hat{N}/M} \text{Ind}_{\hat{M}_x, N \times G}^P \int_{\hat{M}_x} \tau^* \chi^* \otimes \text{Ind}_{\hat{M}_x, N}^G \tau \chi \, d\nu(\tau) \, d\tilde{\mu}(\chi)$$

$$\cong \int_{\hat{N}/M} \int_{\hat{M}_x} \text{Ind}_{\hat{M}_x, N}^P \tau^* \chi^* \otimes L^2(\hat{M}_x N \backslash G; \tau \chi) \, d\nu(\tau) \, d\tilde{\mu}(\chi)$$

$$\cong \int_{\hat{N}/M} \int_{\hat{M}_x} \int_G \text{Ind}_{\hat{M}_x, N}^P \tau^* \chi^* \otimes W_{\chi, \tau}(\pi) \otimes \pi \, d\nu(\tau) \, d\tilde{\mu}(\chi) \, d\tilde{\eta}(\pi).$$

On the other hand

$$L^2(G) \cong \int_G \pi^* |_P \otimes \pi \, d\tilde{\eta}(\pi).$$

Hence $\tilde{\eta}$ is in the measure class of the Plancherel measure, and

$$\pi^* |_P \cong \int_{\hat{N}/M} \int_{\hat{M}_x} \text{Ind}_{\hat{M}_x, N}^P \tau^* \chi^* \otimes W_{\chi, \tau}(\pi) \, d\nu(\tau) \, d\tilde{\mu}(\chi).$$