Research Statement

Raul Gomez

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1 Introduction

The calculation of the space of Whittaker models is one of the main links between number theory and representation theory. A recent result in this area is [Jian, Zhu,Bo]. In [W-86] a multiplicity one result is proved for a class of parabolic subgroups, the very nice parabolic subgroups [W-recent]. Here multiplicity one means the following: if (σ, H_σ) is a finite dimensional representation of M, then

$$\dim Wh_\chi(I^\infty_{P,\sigma,\nu}) = \dim H_\sigma.$$

In section 2 I describe a generalization of this result to infinite dimensional, admissible, smooth, Frechet representations of m, but i restrict P to the case where N is abelian. Besides simplifying some of the calculations this approach has the advantage that all the relevant cases can be described in a nice and uniform manner [G-W]. We distinguish two cases: either the stabilizer of χ in M, M_χ is compact of it isn't-t. The first case is treated in my joint work with Wallach [G-W], in this case the statement on dimension is replaced by an M_χ-equivariant isomorphism induce by some generalized Jacquet integrals which are described in section 2. I look at the second case in [G1]. In this case we need to further refine the statement to an isomorphism

$$Hom_{M_\chi}(H_\sigma, V_\tau) \cong Wh_{\chi,\tau}(I^\infty_{P,\sigma,\nu}).$$

In section 3 I describe an application of this result that I had in mind when I started working on my thesis. In their preprint [S-V] Sakellaridis and Venkatesh describe a way to associate to a symmetric space $9G, H)$ a real reductive group G', and use this construction to conjecture the values of some orbital integrals. Looking at this construction one may wonder if this correspondence extends to a relation between interesting vector spaces associated to (G, H) and G'. In particular we can ask: What is the relation between $L^2(H\backslash H)$ and the representation theory of G'? With this question in mind we turn our attention to the theory of Real Reductive Pairs, in particular we will consider the dual pair $(Sp(m\mathbb{R}) \times O(p, q)) \subset Sp(m(p + q))$.

As an $O(r, s) \times O(p, q)$-module

$$L^2(O(p-r, q-s)\backslash O(p.q)) \cong \int_{O(\hat{p},q)} \int_{O(\hat{r},s)} \tau^* \otimes M_{\tau,r,s}(\pi) \otimes \pi \, d\nu(\tau)\, d\tilde{\eta}(\pi). \qquad (1)$$

On the other hand in [G-Par] I work out the decomposition

$$L^2(N\backslash Sp(m,\mathbb{R});\chi_{r,s}) \tag{2}$$

$$\int_{Sp(\hat{m},\mathbb{R})}\int_{O(\hat{r},s)} W_{\chi_{r,s},\tau}(\pi)\otimes\tau^*\chi^*_{r,s}\otimes\pi\,d\nu(\tau)\,d\mu(\pi) \tag{3}$$

where $W_{\chi_{r,s},\tau}$ is some multiplicity space yet to be determined. The relation between (4) and (3) is

Theorem 1

$$L^2(O(p-r,q-s)\backslash O(p.q)) \cong$$

$$\int_{\hat{G}}\int_{\hat{M_\chi}} W_{\chi,\tau}(\pi)\otimes\tau^*\chi*\otimes\pi\,d\nu(\tau)\,d\mu(\pi) \tag{4}$$

In particular $M_{\tau,r,s}(\Theta(\pi^*)) \cong W_{\chi_{r,s},\tau}(\pi)$ and η is the pullback of μ under the map

$$\Theta: Sp(\hat{m},\mathbb{R}) \longrightarrow O(\hat{p},q),$$

given by the Θ correspondence. Observe that when $m=1$ equation (3) is just the usual Whittaker-Plancherel measure, and in this case

$$W_{\chi_{r,s},\tau}(\pi) \cong Wh_{\chi_{r,s},\tau}(\pi)$$

So it's natural to state the following conjecture

Conjecture 2 *With the notation as above*

$$W_{\chi_{r,s},\tau}(\pi) \cong Wh_{\chi_{r,s},\tau}(\pi).$$

In section 4 we will talk about the state of this conjecture.

2 Bessel Models for General Admissible Induce Representations

Let G be a connected simple Lie group with finite center and let K be maximal compact subgroup. We assume that G/K is Hermitian symmetric of tube type. Up to covering homomorphisms there is a one-to-one correspondence between the set of simple Jordan algebras and the set of Lie groups satisfying this conditions [G-W]. Let $P = MAN$ be a parabolic subgroup, with given Langlands decomposition, such that N is abelian. Let (σ, H_{sigma}) be an admissible, finitely generated, smooth Frechet representation of m, and let ν be a complex valued linear functional on $\mathfrak{a} = Lie(A)$. Let $I^\infty_{P,\sigma,\nu}$ be the smooth representation induced from σ_ν, and let $I^\infty_{M\cap K,\sigma|_{M\cap K}}$ be the representation induced by $\sigma|_{M\cap K}$ from $M\cap K$ to K. Given $f \in I^\infty_{M\cap K,\sigma|_{M\cap K}}$ define

$$f_{P,\sigma,\nu}(namk) = a^{\nu+\rho}\sigma(m)f(k)$$

The map $f \mapsto f_{P,\sigma,\nu}$ defines a K-equivariant linear isomorphism from $I^\infty_{M\cap K,\sigma|_{M\cap K}}$ to $I^\infty_{P,\sigma,\nu}$ [W,vol1]. Let w_M be an element in $N_K(A)$ that conjugates P to $\bar{P}$.

Let χ be a generic character of N [W-86]. Consider the integrals

$$J^{\chi}_{P,\sigma,\nu}(f) = \int_N \chi(n)^{-1} f_{P,\sigma,\nu}(w_M n)\, dn$$

These integrals are called generalized Jacquet integrals and converge absolutely and uniformly on compacta for $\mathrm{Re}\,\nu \gg 0$ [W-86]. Let $M_\chi = \{m \in M \mid \chi(mnm^{-1}) = \chi\}$. We will separate two cases: M_χ is compact or M_χ is not compact.

In my joint work with Wallach [G-W] we looked at the case when M_χ is compact. Let $\mu \in H'_\sigma$ and define $\gamma_\mu(\nu) = \mu \circ J^{\chi}_{P,\sigma,\nu}$. Observe that if Re $\gg 0$ then γ_μ defines a weakly holomorphic map into $(I^{\infty}_{M\cap K,\sigma|_{M\cap K}})'$.

Theorem 3 *Assume that* M_χ *is compact.*

i) γ_μ *extends to a weakly holomorphic map from* $\mathfrak{a}'$ *to* $(I^{\infty}_{M\cap K,\sigma|_{M\cap K}})'$

ii) *Given* $\nu \in \mathfrak{a}'$ *define*

$$\lambda_\mu(f_{P,\sigma,\nu}) = \gamma_\mu(\nu)(f), \qquad f \in I^{\infty}_{M\cap K,\sigma|_{M\cap K}}.$$

Then $\lambda_\mu \in Wh_\chi(I^{\infty}_{P,\sigma,\nu})$ *and the map* $\mu \mapsto \lambda_\mu$ *defines an* M_χ*-equivariant isomorphism between* H'_σ *and* $Wh_\chi(I^{\infty}_{P,\sigma,\nu})$.

This theorem is essentially theorem 12 in [G-W].

In [G1] I look at the case when M_χ is not compact. In this case the above theorem as it is stated is false. However something can still be said about $Wh_\chi(I^{\infty}_{P,\sigma,\nu}$. Assume $M_\chi = {}^\circ M_\chi$ [W,Vol1], and let (τ, V_τ) be an irreducible, admissible infinite dimensional representation of M_χ. Let σ^{w_M} be the twisting of σ by w_M and let $\mu \in Hom(H_{\sigma^{w_M}}, V_\tau)$ (Observe that $\sigma^{w_M}|M\cap K = \sigma$ so this modification was unnecessary in the compact stabilizer case). As in the case were M_χ is compact, define $\gamma_\mu(\nu) = \mu \circ J^{\chi}_{P,\sigma,\nu}$ and observe that if Re $\gg 0$ γ_μ defines a weakly holomorphic map into $Hom(I^{\infty}_{M\cap K,\sigma|_{M\cap K}}, V_{,\tau})$. Let

$$Wh_{\chi,\tau}(I^{\infty}_{P,\sigma,\nu}) = \{\lambda : I^{\infty}_{P,\sigma,\nu} \longrightarrow V_\tau \mid \lambda(\pi(mn)f) = \chi(n)\tau(m)\lambda f\}.$$

Theorem 4 **i)** γ_μ *extends to a weakly holomorphic map from* $\mathfrak{a}'$ *to* $Hom(I^{\infty}_{M\cap K,\sigma|_{M\cap K}}, V_\tau)$.

ii) *Given* $\nu \in \mathfrak{a}'$ *define*

$$\lambda_\mu(f_{P,\sigma,\nu}) = \gamma_\mu(\nu)(f), \qquad f \in I^{\infty}_{M\cap K,\sigma|_{M\cap K}}.$$

Then $\lambda_\mu \in Wh_{\chi,\tau}(I^{\infty}_{P,\sigma,\nu})$ *and the map* $\mu \mapsto \lambda_\mu$ *defines an isomorphism between* $Hom(H_{\sigma^{w_M}}, V_\tau)$ *and* $Wh_{\chi,\tau}(I^{\infty}_{P,\sigma,\nu})$.

In [G1] I prove an equivalent formulation of this result.

3 The Main Example

Consider the dual pair $(Sp(m,\mathbb{R}), O(p,q)) \subset Sp(mn,\mathbb{R})$, with $n = p+q$, and $p \geq q \geq m$. The last condition asserts that we are in the stable case. Let $P = MN$ be the Siegel parabolic subgroup of $Sp(m,\mathbb{R})$. In the theory of the oscillator representation there are very explicit formulas for the action of

$P\times O(p,q)$ on $L^2(\mathbb{R}^{mn})$. Of course this action doesn't extend to a representation of $Sp(m,\mathbb{R})\times O(p,q)$, we would need to go to the double cover to do that, but it does extend to a projective representation of $Sp(m,\mathbb{R})\times O(p,q)$. Using this explicit formulas in [G,Ex] I describe a explicit decomposition of $L^2(\mathbb{R}^{mn})$ as a $P\times O(p,q)$-module. The decomposition is

Proposition 5 *As a $P\times O(p,q)$-module*

$$L^2(\mathbb{R}^{mn})\cong\bigoplus_{r+s=m}\int_{O(\hat{p,q})}\int_{O(\hat{r,s})}M_{\tau,r,s}(\pi)\otimes\mathrm{Ind}^P_{O(r,s)N}\,\tau^*\chi_{r,s}\otimes\pi\,d\nu(\tau)\,d\eta(\pi). \tag{5}$$

On the other hand, the abstract theory of Howe duality establishes that as a $P\times O(p,q)$-module

$$L^2(\mathbb{R}^{mm})\cong\int_{Sp(\hat{m,\mathbb{R}})}\pi|_P\otimes\Theta(\pi)\,d\mu(\pi). \tag{6}$$

Since we are working on the stable range, we know that μ is in the measure class of the Plancherel measure for $Sp(m,\mathbb{R})$. The representation $\Theta(\pi)$ has been determined by the work of Jian-Shu Li [Shu] among others. We are thus left with the problem of decomposing an irreducible tempered representation of $Sp(m,\mathbb{R})$ when restricted to P. In [G-Par] I give the decomposition of π as a representation of P together with other results about representations of parabolic groups that simplify the work of [Wolf], [Howe]. In particular we have

lemma 6 *If π is an irreducible tempered representation of $Sp(m,\mathbb{R})$, then*

$$\pi^*|_P\cong\bigoplus_{r+s=m}\int_{O(\hat{r,s})}W_{\chi_{r,s},\tau}(\pi)\otimes\mathrm{Ind}^P_{O(r,s)N}\,\tau^*\chi^*_{r,s}\,d\nu(\tau). \tag{7}$$

From equation (6) and (7) we obtain

$$L^2(\mathbb{R}^{mn})\cong\bigoplus_{r+s=m}\int_{Sp(\hat{m,\mathbb{R}})}\int_{O(\hat{r,s})}W_{\chi_{r,s},\tau}(\pi)\otimes\mathrm{Ind}^P_{O(r,s)N}\,\tau^*\chi^*_{r,s}\otimes\Theta(\pi^*)\,d\nu(\tau)\,d\mu(\pi). \tag{8}$$

Now from (5) and (8) we obtain the theorem of the introduction. As mentioned before when $m=1$ (**??**) is just the Plancherel-Whittaker theorem explicitly described in [W,vol2]. In that decomposition we have that $W_{\chi_{r,s},\tau}(\pi)=Wh_{\chi_{r,s},\tau}(\pi)$, it is therefore natural to formulate the following conjecture.

Conjecture 7 *With notation as above*

$$W_{\chi_{r,s},\tau}(\pi)=Wh_{\chi_{r,s},\tau}(\pi)$$

For all r,s and all τ,π tempered and irreducible.

An important step in the direction of proving the conjecture is the calculation of the space $Wh_{\chi_{r,s}}(\pi)$ where π is an induced representation which is done in [G-W] when $M_{\chi_{r,s}}\cong O(r,s)$ is compact, i.e., $r=0$ or $s=0$, and in [G-1] in all the other cases. Using the results in [G-W] and following the ideas given in chapter 15 of [W,vol2] I have been able to calculate the Bessel-Plancherel measure of $L^2(N\backslash G;\chi)$ in the case where M_χ is compact. The calculations and the explicit Plancherel measure and intertwiners can be found in [G-2] and prove the conjecture in the compact stabilizer case. The full conjecture should be proved in a similar way, but first some complications of functional analysis nature should be solved

4 Further Research

Besides finishing the calculations of the measure of $L^2(N\backslash G;\chi)$ where χ is an arbitrary character, there are some other directions on which the results can be improved.

In [W-recent] there is a classification of nice and very nice parabolic subgroups. In [G-W] we restricted our attention to the abelian unipotent case to simplify the exposition given in [W-86], and because in that paper a nice description of all the pertinent groups was given. It would be interesting to check if the calculations of $Wh_{\chi,\tau}(\pi)$ can be applied to all very nice parabolic subgroups. Besides being a first step in the calculation of the Whittaker-Plancherel measure this spaces of Whittaker functionals are one of the main links between Number Theory and Representation Theory. [Jian,zhu,bo].

Another interesting problems is the calculation of $Wh_\chi(\pi)$ for χ not a character but a general irreducible unitary representation of N. A good example is when P is the Heisenberg parabolic, and ξ_χ is the irreducible representation associated with the central character χ. Some work and explicit calculations have been given in [Narita] and it's tempting to try to use our approach to this kind of results.

Another natural extension of this result is to look at other reductive pairs, not only to other dual pairs in the oscillator representation, but also to minimal representations of some exceptional groups, for example the representations described in [Gan-Savin]. Of particular interest are the case where on of the groups in the dual pair has rank 1 or 2, as this could lead to a case by base prove of conjecture ?? of [S-V].

The results given here should have equivalents in the p-adic and automorphic settings. In this case some arguments (like the holomorphic continuation of the Jacquet integrals given in [G-W]) should be adapted. But following the ideas given in [Holomorphic continuation of Jacquet integrals for minimal parabolic on p-adic fields] the corresponding results may be found.

As we have mentioned before, in [S-V] an algorithm is given to associate to a certain symmetric space (G,H), of rank l, a real reductive group G' of rank l. It's expected that we can use the representation theory of this group G' to understand certain interesting spaces associated to the symmetric pair (G,H). In this sense this thesis is an example of the results one is expected to obtain, in our case we have associated the space $L^2(O(p-r,q-s)\backslash O(p,q))$ and the space of $O(p-r,q-s)$-invariant linear functionals of a smooth irreducible representation π of $O(p,q)$ satisfying certain properties to the space $L^2(N\backslash Sp(m,\mathbb{R});\chi)$ of Whittaker functions and to the space of Whittaker functionals of the representation $\Theta(\pi)$ associated to π. As was mentioned before using the theory of dual pairs one may obtain other examples of this kind of correspondence. it-s a long term goal to try to understand this correspondences in an intrinsic way, without having to resort to the dual pair construction.