PE 1

Math 2220
Prelim 1
February 22nd, 2011

Name: ____________________

TA's name: ____________________

Discussion: ____________________

**INSTRUCTIONS — READ THIS NOW**

- This test has **6** problems on 9 pages (counting this one and two blank pages at the end) worth a total of 100 points.
- Write your name, your TA's name, and your discussion section number **right now**.
- Show your work/explanation. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to clearly label your work.
- This is a *closed-book* test. Notes, books, "cheat sheets", cell phones, and personal audio players are NOT allowed. Calculators are neither needed nor permitted.
- This is a **90** minute test.

| OFFICIAL USE ONLY | | |
|---|---|---|
| 1. | ____ | /16 |
| 2. | ____ | /16 |
| 3. | ____ | /21 |
| 4. | ____ | /21 |
| 5. | ____ | /16 |
| 6. | ____ | /10 |
| Total: | ____ | /100 |

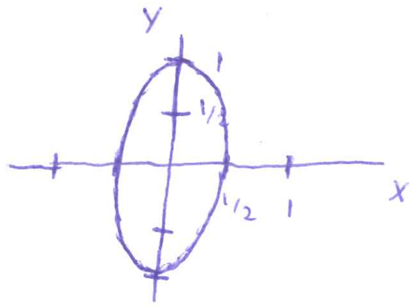
**Question 1 (16pts):** Let $f(x, y) = 2x^2 + y^2$ and $g(x, y) = y^2/x$.

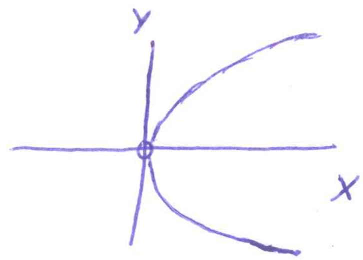
(a) Sketch the level curves $f(x, y) = 1$ and $g(x, y) = 1$.

(b) Show that the point $(1/2, 1/\sqrt{2})$ is in the intersection of the two level curves from part (a).

(c) Compute $\nabla f(1/2, 1/\sqrt{2}) \cdot \nabla g(1/2, 1/\sqrt{2})$. What does this mean geometrically about the level curves in part (a)?

a) $f(x,y)=1 \Rightarrow 2x^2+y^2=1$ $\qquad$ $g(x,y)=1 \Rightarrow y^2/x = 1 \Rightarrow y^2 = x \quad (x \neq 0)$




b) $f(1/2, 1/\sqrt{2}) = 2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= 2/4 + 1/2 = 1$

$g(1/2, 1/\sqrt{2}) = (1/\sqrt{2})^2 / (1/2)$

$= \frac{2}{(\sqrt{2})^2} = 1$

Hence $P = (1/2, 1/\sqrt{2})$ is in both level curves.

c) $\frac{\partial f}{\partial x}(x,y) = 4x$, $\frac{\partial f}{\partial y} = 2y$, $\qquad \frac{\partial g}{\partial x}(x,y) = -\frac{y^2}{x^2}$, $\frac{\partial g}{\partial y}(x,y) = 2y/x$

Hence

$\nabla f(1/2, 1/\sqrt{2}) = (4/2, 2/\sqrt{2})$

$= (2, 2/\sqrt{2})$

$\nabla g(1/2, 1/\sqrt{2}) = \left(-\frac{(1/\sqrt{2})^2}{(1/2)^2}, \frac{2(1/\sqrt{2})}{1/2}\right)$

$= (-2, 4/\sqrt{2})$

Therefore,

$\nabla f(1/2, 1/\sqrt{2}) \cdot \nabla g(1/2, 1/\sqrt{2}) = (2, 2/\sqrt{2}) \cdot (-2, 4/\sqrt{2})$

$= -4 + \frac{8}{2} = 0.$

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Last part does not apply

**Question 2 (16pts):** Consider the following function $f(x,y) = \frac{xy^2}{x^3+2y^3}$.

(a) Find the limit $(x,y) \to (0,0)$ of the function $f(x,y)$ *along a line* $y = x$.

(b) Does the following limit exists? Justify your answer:

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

a) When $y = x$, we have that

$$f(x,y) = f(x,x) = \frac{x(x)^2}{x^3+2x^3} = \frac{1}{3}.$$

Hence, the limit $(x,y) \to (0,0)$ of the function $f$ along the line $y = x$ is

$$\lim_{x\to 0} f(x,x) = \lim_{x \to 0} \frac{1}{3} = \frac{1}{3}.$$

b) No. Along the line $y = -x$, the limit is

$$\lim_{x\to 0} f(x,-x) = \lim_{x\to 0} \frac{x(-x)^2}{x^3+2(-x)^3} = \lim_{x\to 0} -1 = -1.$$

Since this limit is different than the limit along the line $y = x$, we conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{DNE.}$$

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**Question 3 (21pts):** Consider the function $w = F(x, y, z) = x^2 - y^2 - z$ of three variables.

(a) Find the direction of fastest increase at $(1, 1, 1)$ in $\mathbb{R}^3$ and the rate of change of $F$ in that direction (directional derivative).

(b) Find the equation of the tangent plane $H_{(a,b,c)}$ to the level surface at $(a, b, c)$.

(c) Consider the tangent plane $H_{(1,1,1)}$ to the level surface at $(1, 1, 1)$. Find a point $(a, b, 1)$ such that the tangent plane $H_{(a,b,1)}$ to the level surface at $(a, b, 1)$ is perpendicular to $H_{(1,1,1)}$.

NA

**Question 4 (21pts):** Let $z = f(x, y) = x^2 + (y-1)^2$.

(a) Classify all the critical points in the open disk given by $x^2 + y^2 < 4$ to local min/max and saddle points.

(b) Find the absolute max/min of $f(x, y)$ over the circle given by $x^2 + y^2 = 4$.

(c) Find the absolute max/min of $f(x, y)$ over the closed disk given by $x^2 + y^2 \leq 4$.

NA

**Question 5 (16pts):** Consider the surface given by $F(x, y, z) = \sin z + xy - 2 = 0$ and a point $p := (2, 1, 0)$ on it.

(a) We would like to say that in a neighborhood of $p$, the surface is a graph of a function $z = g(x, y)$ by using the implicit function theorem. Which is the most relevant reasoning for that?

i. Because the gradient $\nabla F|_{(2,1,0)}$ is not a zero vector.

ii. Because $F_z(2, 1, 0)$ is not zero.

iii. Because $F_z$ is differentiable at $(2, 1, 0)$.

(b) Find the linear approximation of $z = g(x, y)$ at $p$ without solving the equation $\sin z + xy - 1 = 0$ for $z$.

NA

**Question 6 (10pts):** Suppose that we need to make a box with the maximum volume. The only constraint is that the sum of the length, width and height of the box must be 18 inches.

(a) What should the dimension of the box be to maximize the volume? Find the candidate by using Lagrange Multiplier method.

(b) Give an explanation why the dimensions you found in question (a) would give actually the maximum volume.

NA

**STOP. THIS IS THE LAST PAGE OF PROBLEMS.**