PE 2

Math 2220 Exam 1

Tuesday, February 21, 2012

Name: ______________________________

Show all work and explain all answers except as noted.

1. a. Find the equation of the tangent plane to the surface defined by

$$\ln(x^2 + z^4 - 1) - x^2(y - 2) - z = 0$$

at the point $(1, 3, -1)$.

NA

b. Find all values of (a, b) for which the tangent plane to $z = x^2y - xy^2 - x + y + xy$ at $(x, y) = (a, b)$ parallel to the plane $2x - 2y + 2z = 5$.

NA

2. Find all local maxima, local minima, and saddle points of $f(x,y) = x^4 - xy + \frac{1}{4}y^4$. (You must indicate which are local maxima, which are local minima, and which are saddle points.)

NA

3. Define $f(x,y) = xy/(x^2+y^2)$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Determine the set of all (x,y) such that $\frac{\partial f}{\partial x}$ is defined and find the value when possible. Is $\frac{\partial f}{\partial x}$ continuous?

If $(x,y) \neq (0,0)$, then

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^3 - x^2y}{(x^2+y^2)^2} \qquad (1)$$

On the other hand, using the definition of partial derivative, we have that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h(0)}{(h^2+0^2)} - 0}{h} = \lim_{h \to 0} 0 = 0.$$

Hence, $\frac{\partial f}{\partial x}$ is defined everywhere.

From eq. (1) it's clear that $\frac{\partial f}{\partial x}(x,y)$ is continuous when $(x,y) \neq (0,0)$. To check if $\frac{\partial f}{\partial x}$ is cont. at $(0,0)$ we calculate the limit

$$\lim_{(x,y) \to (0,0)} \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

Using polar coordinates, this limit becomes

$$\lim_{r \to 0} \frac{r^3(\sin^3\theta - \cos^2\theta \sin\theta)}{r^4} = \lim_{r \to 0} \frac{\sin^3\theta - \cos^2\theta\sin\theta}{r}$$

but the above limit DNE. We conclude that $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$.

4. Find the absolute maximum and minimum value of the function $f(x, y) = x^2 + y^2 + 2y$ on the set $D = \{(x, y) : x^2 + y^2 \leq 2\}$.

NA

5. a. Define

$$\mathbf{r}(t) = \langle \frac{1}{2}\cos(2t) + \frac{1}{2}, \frac{1}{2}\sin(2t), \sin(t) \rangle.$$

Compute the velocity vector as a function of t.

$$r'(t) = \langle -\sin 2t, \cos 2t, \cos t \rangle$$

b. Find the equation of the line tangent to the curve parametrized by $\mathbf{r}(t)$ at $(1, 0, 0)$.

Observe that

$$r(0) = \langle \frac{1}{2} + \frac{1}{2}, \frac{1}{2}(0), 0 \rangle = (1, 0, 0).$$

Using part a), we have that

$$r'(0) = \langle 0, 1, 1 \rangle.$$

Hence, the eq. that we are looking for is

$$l(t) = r(0) + t\, r'(0) = (1, 0, 0) + t(0, 1, 1) = (1, t, t).$$

c. Show that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all t. (The following identities may be helpful: $\sin(2t) = 2\sin(t)\cos(t)$ and $\cos(2t) = 2\cos^2(t) - 1 = \cos^2(t) - \sin^2(t)$.)

By part a),

$$r(t) \cdot r'(t) = \left(\frac{1}{2}\cos 2t + \frac{1}{2}\right)(-\sin 2t) + \frac{1}{2}\cos 2t \sin 2t + \sin t \cos t$$

$$= -\frac{1}{2}\cancel{\cos 2t \sin 2t} - \frac{1}{2}\sin 2t + \frac{1}{2}\cancel{\cos 2t \sin 2t} + \sin t \cos t$$

$$= \sin t \cos t - \frac{1}{2}\sin 2t = 0$$

where we are using the identity

$$\sin 2t = 2 \sin t \cos t.$$

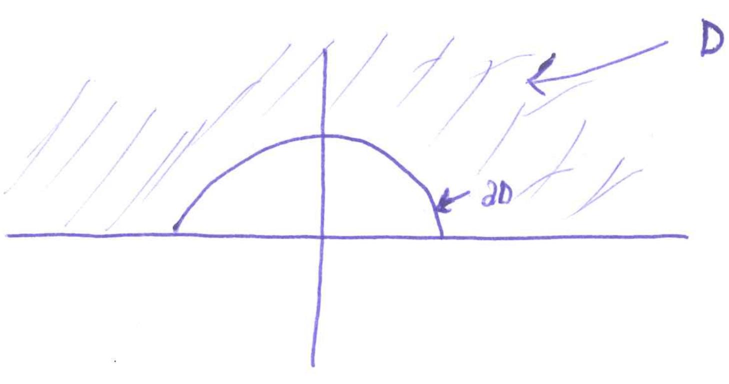
6. a. Show that there is a local solution $z = f(x, y)$ to $xy^2z^5 - 2x^3yz + 4x^2y = 3$ at the point $(1, 1, 1)$. Compute $\frac{\partial z}{\partial x}|_{(1,1,1)}$.

NA

b. Let U be a subset of $\mathbb{R}^n$. Define what is meant by the phrase "U is open."

By def, U is open if for every pt $x \in U$ there exists $r > 0$ such that the open ball $D_r(x) \subset U$.

c. If $D = \{(x, y) : (x^2 + y^2 > 1) \text{ and } (y > 0)\}$, what is the boundary of D? Your answer should both contain a sketch and a clear description of the set. You do not have to justify your answer.



$\partial D = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } y \geq 0 \}$