FALL 2012 PRELIMINARY EXAM 1 SOLUTIONS (PE3)

1. Find a vector normal to the surface

$$x^2 - xy + yz - 3 = 0$$

at the point $(1, 2, 2)$.

Sol NA

2. Find the area of the triangle formed by the points:

$$\text{A } (1,1,1), \quad \text{B } (1,2,3) \quad \text{C } (2,3,1)$$

Sol Let $\mathbf{v} = \overrightarrow{AB} = (0, 1, 2)$ and $\mathbf{w} = \overrightarrow{AC} = (1, 2, 0)$. Then the area of the parallelogram generated by the vectors $\mathbf{v}$ and $\mathbf{w}$ is given by $\|\mathbf{v} \times \mathbf{w}\|$. Since

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = \mathbf{i}(-4) - \mathbf{j}(-2) + \mathbf{k}(-1) = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

we have that

$$\|\mathbf{v} \times \mathbf{w}\| = \sqrt{16 + 4 + 1} = \sqrt{21}.$$

Since the area of the triangle ABC is half the area of the above parallelogram, we conclude that

$$\text{Area } \Delta\text{ABC} = \frac{\|v \times w\|}{2} = \frac{\sqrt{21}}{2}.$$

3. Find the equation of the plane which:
(i) contains the line

$$\mathbf{r} = (x, y, z) = (1, 2, 3) + t(4, 5, 6)$$

and (ii) is perpendicular to the plane

$$3x + 2y + z = 1$$

Sol Let

$$\mathbf{n} = (3, 2, 1), \qquad \mathbf{v} = (4, 5, 6).$$

Observe that the vector $\mathbf{n} = (3, 2, 1)$ is normal (that is, perpendicular) to the plane $3x+2y+z = 1$. Since the plane we are looking for (let's call it $\mathcal{P}$) should also be perpendicular to the plane $3x + 2y + z = 1$, we conclude that the vector with components $\mathbf{n} = (3, 2, 1)$ and starting point $(1, 2, 3)$ should be contained in our plane $\mathcal{P}$. On the other hand, the vector with components $\mathbf{v} = (4, 5, 6)$ and starting point $(1, 2, 3)$ should also be contained in our plane $\mathcal{P}$, because the whole line $\mathbf{r}(t) = (1, 2, 3) + t(4, 5, 6)$ is contained in our plane. Hence the vector

$$\mathbf{n} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 4 & 5 & 6 \end{vmatrix} = \mathbf{i}(12 - 5) - \mathbf{j}(18 - 4) + \mathbf{k}(15 - 8) = 7\mathbf{i} - 14\mathbf{j} + 7\mathbf{k}$$

is perpendicular to our plane $\mathcal{P}$. Since we already know that the point $(1, 2, 3)$ belongs to this plane, we deduce that

$$7x - 14y + 7z = (7, -14, 7) \cdot (1, 2, 3) = 7 - 28 + 21 = 0$$

is the equation of the plane $\mathcal{P}$.