Math 2220
Prelim 1
February 19th 2013 Duration - 1.5 hours

Last Name: ____________________

Given Name: ____________________

NetID : ____________________

Please circle your instructor/lecture section and your TA/discussion section.

Bruce Fontaine	MWF 10:10am - 11:00am
Allen Knutson	MWF 12:20pm - 1:10pm

Zhexiu Tu	W 1:25pm - 2:15pm (MRL 106)
Chenxi Wu	W 2:30pm - 3:20pm (RCK 102)
Zhexiu Tu	W 2:30pm - 3:20pm (MRL 106)
Zhexiu Tu	W 3:35pm - 4:25pm (MRL 106)
Chenxi Wu	W 3:35pm - 4:25pm (RCK 102)

Question	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total:	100	

INSTRUCTIONS

1. This exam has 6 questions worth 100 points on 7 pages (including this one).
2. Please enter your name and circle your lecture/discussion sections now.
3. Please show your work and when possible indicate your final answer by putting it in $\boxed{\text{a box.}}$
4. You are allowed a single, one-sided letter sized page of notes in the exam. No other aids are allowed (i.e. no calculators).

1. (15 points) Define a function

$$f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

a) Determine the set of points where f is continuous.
b) Compute the partial derivatives where they exist.
c) Are the partial derivatives continuous at all points in $\mathbb{R}^2$?

Solution: a) f is continuous away from $(0,0)$ as it is the quotient of a continuous function by a non zero continuous function there. The only thing that needs to be checked is the behavior at $(0,0)$. Checking along the path $y = 0$, $\lim_{x\to 0} f(x,0) = 1$. On ther other hand for $x = 0$, $\lim_{y\to 0} f(0,y) = 0$. Thus f is not continuous at 0.
b) Away from $(0,0)$ the partials are $\frac{\partial f}{\partial x} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$ and $\frac{\partial f}{\partial y} = -\frac{yx}{(x^2+y^2)^{\frac{3}{2}}}$. We only need to check the existence of the partials at $(0,0)$ and for this we must use the limit definition of the partial

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-0}{h} = \lim_{h\to 0} \frac{1}{h} = \infty,$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Thus the y partial exists everywhere while the x partial does not exist at $(0,0)$.
c) The partials cannot be continuous on all of $\mathbb{R}^2$ since this would mean that f is differentiable there and thus is continuous!

2. (15 points) Let $f : U \to \mathbb{R}$ where f is a continuous function and $U \subset \mathbb{R}^n$ is an open set.
Let $\boldsymbol{x} \in U$ such that $f(\boldsymbol{x}) > 0$.
Show that there is a nonempty open set $V \subset \mathbb{R}^n$ such that $f(\boldsymbol{y}) > 0$ for all $\boldsymbol{y} \in V$.

Solution: Let $r = f(\boldsymbol{x})$, then apply the definition of continuity at $\boldsymbol{x}$ to f. For the ball of radius r around $f(\boldsymbol{x})$, that is the interval $(0, 2f(\boldsymbol{x}))$, continuity states there exists a ball V of radius $s > 0$ around $\boldsymbol{x}$ such that $f(V) \subset (0, 2f(\boldsymbol{x}))$. Thus f is positive on V.

3. (15 points) A subset $U \subset \mathbb{R}^n$ is called **path-connected** if for every $\boldsymbol{x}, \boldsymbol{y} \in U$, there exists a smooth function $\boldsymbol{c} : \mathbb{R} \to \mathbb{R}^n$ (the "path") such that $\boldsymbol{c}(0) = \boldsymbol{x}$ and $\boldsymbol{c}(1) = \boldsymbol{y}$.
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function on an open path-connected set U. If $\nabla f(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in U$, then show that f is constant.
Hint: show that $f(\boldsymbol{x}) = f(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in U$.

Solution: We choose an $\boldsymbol{x} \in U$ and we will show that f takes the constant value $f(\boldsymbol{x})$ on U. For any point $\boldsymbol{y} \in U$, we have a path $\boldsymbol{c} : \mathbb{R} \to \mathbb{R}^n$ with $\boldsymbol{c}(0) = \boldsymbol{x}$ and $\boldsymbol{c}(1) = \boldsymbol{y}$. Consider the function $f \circ \boldsymbol{c}$, it has derivative 0, since via the chain rule we have $\frac{d}{dt}(f \circ \boldsymbol{c})(t) = \nabla f(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) = 0$. Thus we know that $f \circ \boldsymbol{c}$ is constant, so $(f \circ \boldsymbol{c})(0) = (f \circ \boldsymbol{c})(1)$, hence $f(\boldsymbol{x}) = f(\boldsymbol{y})$ as needed.

4. (15 points) Find the equation of the tangent plane at the point $(1, 0, 0)$ to the surface $z+1 = xe^y \cos(z)$.

Solution: We begin by seeing the surface as a level set $1 = xe^y \cos(z) - z$. Then taking the gradient of the resulting function we have $\nabla(xe^y \cos(z) - z) = (e^y \cos(z), xe^y \cos(z), -xe^y \sin(z) - 1)$. Evaluating at $(1, 0, 0)$ we have $(1, 1, -1)$. Since the gradient is non zero, the equation is given by $(1, 1, -1) \cdot ((x, y, z) - (1, 0, 0)) = 0$, or $x + y - z = 1$.

5. (20 points) Consider the function

$$f(x,y) = \begin{cases} \quad 1 \text{if } x > 0 \text{ and } y > \sqrt{x} \\ 0 \quad \text{otherwise.} \end{cases}$$

a) Along each line $y = mx$ through the origin, of slope m, calculate the limit of $f(x,y)$ as $(x,y) \to (0,0)$.
b) Is $f(x,y)$ continuous at $(0,0)$? Prove or disprove.
Hint: it may help to draw the regions where $f(x,y)$ is 0 and 1.

Solution: a) This limit is $\lim_{x\to 0} f(x, mx)$. To understand this limit we need to understand the behavior of f along this line. We will show that this limit is always 0, to do this, it is sufficient to show that it is zero on some open set around 0.

Now, $f(x, mx)$ is 0 when $x \le 0$ or when $mx \le \sqrt{x}$. Simplifying the second condition gives $m\sqrt{x} \le 1$. We now break it down via cases based on m: if $m = 0$ then we have $f(x, 0) = 0$ for any x. Similarly when $m < 0$, then the second condition becomes $\sqrt{x} \ge \frac{1}{m}$, which is satisfied when $x \le 0$ and when paired with the first condition, we have $f(x, mx) = 0$ for all x when $m < 0$. If $m > 0$, then $\sqrt{x} \le \frac{1}{m}$ or $x \le \frac{1}{m^2}$, that is, so long as $x \le \frac{1}{m^2}$ then $f(x, mx) = 0$, but the set of x satisfying this condition is open and includes 0, so $\lim_{x\to 0} f(x, mx) = 0$ for any m.
b) $f(x,y)$ is discontinuous at 0, there are two easy ways to show this: If f was continuous then $f \circ \phi$ is continuous for any continuous path ϕ. Take $\phi(t) = (t, 2\sqrt(t))$, then

$$(f \circ \phi)(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

This function is has a jump discontinuity at 1.

Alternatively, since we know the value of $f(0,0) = 0$, we just need to exhibit points (x,y) arbitrarily close to $(0,0)$ with $f(x,y) = 1$. Points of the form $f(x, 2\sqrt(x))$ would suffice.

6. (20 points) A rectangular box without a lid (i.e. it only has 5 sides) is constructed from 27 m^2 of cardboard. We want to find the maximum possible volume.
a) First, set up this problem as a maximization problem you can solve.
b) Then, solve.

Solution: a) Let l, w and h be the 3 dimensions of the box. Then the area of cardboard used is $2lh + 2wh + lw = 27$, on the other hand the volume is lwh. Solving the first equation for h, we get $h = \frac{27-lw}{2l+2w}$. Since l and w are both positive, this is always well defined, so the volume of the box is $\frac{27lw-(lw)^2}{2l+2w}$.
b) Now we need to find the critical points of the function and find the maximum value. Taking the gradient we get

$$\left(\frac{w^2(-27+l^2+2lw)}{2(l+w)^2}, \frac{l^2(-27+w^2+2lw)}{2(l+w)^2}\right).$$

Setting this to zero, the critical points must satisfy $w^2(-27+l^2+2lw) = 0$ and $l^2(-27+w^2+2lw)$. We immediately discard the solutions $w = 0$ or $l = 0$ since these would result in zero volume. We get $l^2 + 2lw - 27 = 0$ and $w^2 + 2lw - 27$, subtracting the two equations gives $l^2 - w^2 = 0$, so that either $l = w$ or $l = -w$. Since they are both positive we have have $l = w$, this gives $l^2 - 2l^2 - 27 = 0$, or $l = \pm 3$. Thus the max occurs at $w = l = 3$ and is $\frac{27*9+81}{12} = \frac{27}{2}$.