Math 2220
Prelim 2
March 28th 2013 Duration - 1.5 hours

Last Name: ______________________

Given Name: ______________________

NetID : ______________________

Please circle your instructor/lecture section and your TA/discussion section.

Bruce Fontaine	MWF 10:10am - 11:00am
Allen Knutson	MWF 12:20pm - 1:10pm

Zhexiu Tu	W 1:25pm - 2:15pm (MRL 106)
Chenxi Wu	W 2:30pm - 3:20pm (RCK 102)
Zhexiu Tu	W 2:30pm - 3:20pm (MRL 106)
Zhexiu Tu	W 3:35pm - 4:25pm (MRL 106)
Chenxi Wu	W 3:35pm - 4:25pm (RCK 102)

Question	Points	Score
1	20	
2	10	
3	15	
4	20	
5	20	
Total:	85	

INSTRUCTIONS

1. This exam has 5 questions worth 85 points on 6 pages (including this one).
2. Please enter your name and circle your lecture/discussion sections now.
3. Please show your work and when possible indicate your final answer by putting it in $\boxed{\text{a box.}}$
4. You are allowed a single, one-sided letter sized page of notes in the exam. No other aids are allowed (i.e. no calculators).

1. (20 points)

   (a) Find the critical points of $F(x, y, z) = \log x + \log y + 3 \log z$ on $x^2 + y^2 + z^2 = 5r^2$ where $x, y, z > 0$.

   (b) What kind are they? i.e. max, min, other?

Solution: Let $G(x, y, z) = x^2+y^2+z^2$, then $\nabla F(x, y, z) = \left(\frac{1}{x}, \frac{1}{y}, \frac{3}{z}\right)$ and $\nabla G(x, y, z) = (2x, 2y, 2z)$. The Lagrange multipliers equation is

$$\left(\frac{1}{x}, \frac{1}{y}, \frac{3}{z}\right) = \lambda(2x, 2y, 2z).$$

This results in the 3 equations $1 = \lambda 2x^2$, $1 = \lambda 2y^2$ and $3 = \lambda 2z^2$. Eliminating λ from the first two gives $x^2 = y^2$ and since x and y are both positive we have $x = y$. Similarly we get $\sqrt{3}x = z$. Substituting these into the level set $G(x, y, z) = 5r^2$, we get $G(x, x, \sqrt{3}x) = 5r^2$, or $5x^2 = 5r^2$ i.e. $x = r$.

Thus the point $(r, r, \sqrt{3}r)$ is a critical point. Since $x, y, z \leq \sqrt{5}r$ and log is an increasing function $F(x, y, z)$ is bounded above by $5 \log \sqrt{5}r$ and as x, y or z approach 0 (i.e. the boundary of the surface), $F(x, y, z)$ decreases towards $-\infty$. Thus it has a global max and it must be $(r, r, \sqrt{3}r)$.

Alternatively you can use the bordered Hessian. To do this we construct the axillary function $h(x, y, z) = logx + \log y + 3 \log z - \lambda(x^2 + y^2 + z^2)$. Then taking various derivatives gives:

$$\begin{vmatrix} 0 & -2x & -2y & -2z \\ -2x & -\frac{1}{x^2} - 2\lambda & 0 & 0 \\ -2y & 0 & -\frac{1}{y^2} - 2\lambda & 0 \\ -2z & 0 & 0 & -\frac{3}{z^2} - 2\lambda \end{vmatrix}$$

Evaluating this at $(r, r, \sqrt{3}r)$ and $\lambda = \frac{1}{2r^2}$ gives

$$\begin{vmatrix} 0 & -2r & -2r & -2\sqrt{3}r \\ -2r & -\frac{2}{r^2} & 0 & 0 \\ -2r & 0 & -\frac{2}{r^2} & 0 \\ -2\sqrt{3}r & 0 & 0 & -\frac{2}{r^2} \end{vmatrix}.$$

The full determinant is $-\frac{80}{r^2}$ and the determinant of the upper left 3 by 3 submatrix is 16. Since 16 is positive and $-\frac{80}{r^2}$ is negative, this point is a local maximum.

2. (10 points) Find the arc length of $\boldsymbol{c}(t) = (t, 3t^2, 6t^3)$ from $t = 0$ to $t = 2$.

Solution: First we calculate $\boldsymbol{c}'(t) = (1, 6t, 18t^2)$ and

$$||\boldsymbol{c}'(t)|| = \sqrt{1 + 36t^2 + 18^2t^4} = \sqrt{(1 + 18t^2)^2} = 1 + 18t^2.$$

Then the arc length is the integral

$$\int_0^2 (1 + 18t^2)dt = t + 6t^3\Big|_0^2 = 2 + 48 = 50.$$

3. (15 points) Let

$$f(x) = \begin{cases} 5 & \text{if } x = \sqrt{3} \\ 0 & \text{otherwise.} \end{cases}$$

Show (which means **prove**) directly from the definition of integrability that f is integrable on $[0, 2]$ and compute the integral.

Solution: Recall that we defined integrability by dividing the interval $[0, 2]$ into n equation parts of length $\frac{2}{n}$. Then in each interval we chose a point c_i and we calculated the Riemann sum

$$\sum_{i=1}^{n} f(c_i) \frac{2}{n}.$$

Now the i-th interval is $[\frac{i-1}{n}, \frac{i}{n}]$ and since the end points are rationals, $\sqrt{3}$ can only be in one interval, hence in our choices of c_i, at most 1 can be $\sqrt{3}$. So this means that

$$0 \leq \sum_{i=1}^{n} f(c_i) \frac{2}{n} \leq \frac{5 * 2}{n}.$$

Taking the limit as $n \to \infty$ show us that $\lim_{n\to\infty} S_n$ converges to 0 independently of the choice of the c_i. Thus the function is integrable and the integral is 0.

4. (20 points) We know that a vector field $\boldsymbol{F}$ on $\mathbb{R}^3$ cannot be the gradient of a function f unless $\operatorname{curl} \boldsymbol{F} = 0$. On the other hand there may exist some nonzero $\mu : \mathbb{R}^3 \to \mathbb{R}$ such $\mu\boldsymbol{F}$ is a gradient vector field. Show that $\boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{F} = 0$ in this case.

Solution: The simplest solution is to recall the 'product rule' for $\mu\boldsymbol{F}$ under curl: $\operatorname{curl} \mu\boldsymbol{F} = \mu \operatorname{curl} \boldsymbol{F} + (\nabla\mu) \times \boldsymbol{F}$. Since $\mu\boldsymbol{F}$ is the gradient of a function, the previous statement is equal to 0. Dotting both sides with $\boldsymbol{F}$ gives

$$0 = \mu\boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{F} + \boldsymbol{F} \cdot (\nabla\mu) \times \boldsymbol{F} = \mu\boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{F}.$$

Since μ is nonzero, we get $\boldsymbol{F} \cdot \operatorname{curl} \boldsymbol{F} = 0$.

This can also be done via direct calculation: Since $\mu\boldsymbol{F}$ is a gradient vector field, $\operatorname{curl}(\mu\boldsymbol{F}) = 0$. Looking at the first component we get $\mu((F_2)_z - (F_3)_y) + \mu_z F_2 - \mu_y F_3 = 0$, thus we can isolate

$$(F_2)_z - (F_3)_y = \frac{-\mu_z F_2 + \mu_y F_3}{\mu}.$$

We get similar formulas for the other two components and altogether the left hand sides are $\operatorname{curl} \boldsymbol{F}$. One can check that dotting the vector formed of the right hand sides with $\boldsymbol{F}$ gives 0.

5. (20 points) Evaluate the integral

$$\int_0^2 \int_{(y/2)-1}^0 \ln(5+2x+x^2)\,dx\,dy.$$

Solution: We change the order of the integral. The region of integration is the triangle whose verticies are the points $(0,0)$, $(-1,0)$ and $(0,2)$. On this region $-1 \leq x \leq 0$, and $0 < 5+2x+x^2$, so the integrand is continuous and we can switch the order of integration. Switching the order the integral becomes

$$\int_{-1}^0 \int_0^{2+2x} \ln(5+2x+x^2)\,dy\,dx = \int_{-1}^0 (2+2x)\ln(5+2x+x^2)\,dx.$$

Using the substitution $u = 5+2x+x^2$, this integral transforms to

$$\int_4^5 \ln(u)\,du = u\ln u - u|_4^5 = 5\ln 5 - 5 - 4\ln 4 + 4 = 5\ln 5 - 4\ln 4 - 1.$$