MATH2220 Spring 2011 Prelim 2 Solution

(1) (a)

$$\int_0^a \int_y^{2y} y^2 e^{y^2} dxdy, \qquad \int_0^a \int_{x/2} xy^2 e^{y^2} dxdy + \int_a^{2a} \int_{x/2}^a y^2 e^{y^2} dxdy$$

(b)

$$\int_0^a \int_y^{2y} y^2 e^{y^2} dxdy = \int_0^a y^2 e^{y^2} \int_y^{2y} dxdy = \int_0^a y^2 e^{y^2} [x]_y^{2y} dy = \int_0^a y^3 e^{y^2} dy$$

$$= \frac{1}{2} \int_0^{a^2} ue^u du = \frac{1}{2} \left\{ [ue^u]_0^{a^2} - \int_0^{a^2} e^u du \right\} = \frac{1}{2} \left\{ a^2 e^{a^2} - e^{a^2} + 1 \right\}$$

where we use the substitution $u = y^2$ and $du = 2ydy$.

(2) (a) Sketch is omitted, but the region is given by

$$D = \{0 \le x \le \ln 3, 0 \le z \le 1, \sqrt[3]{z} \le y \le 1\} = \{0 \le x \le \ln 3, 0 \le y \le 1, 0 \le z \le y^3\}.$$

(b)

$$\int_0^{\ln 3} \int_0^1 \int_0^{y^3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dzdydx = \pi \int_0^{\ln 3} e^{2x} dx \cdot \int_0^1 \frac{\sin(\pi y^2)}{y^2} \int_0^{y^3} dzdy$$

$$= \pi [e^{2x}/2]_0^{\ln 3} \cdot \int_0^1 \frac{\sin(\pi y^2)}{y^2} y^3 dy = \frac{(e^{2\ln 3} - 1)}{2} \int_0^1 \pi y \sin(\pi y^2) dy$$

$$= \frac{(9-1)}{2} \int_0^{\pi} \frac{1}{2} \sin u du = 2[-\cos u]_0^{\pi} = 4$$

where we used the substitution $u = \pi y^2$ and $\frac{1}{2} du = \pi y dy$.

(3) We want to compute the area inside the circle $x^2 + y^2 = 8$ and outside $x^2 + (y-2)^2 = 4$. We look at each quadrant separately: in each 2nd and 3rd quadrants, we have 2π so in total 4π (half of the disk $x^2 + y^2 \le 8$). The areas in 1st and 4th quadrants are the same so we compute what we have in the 1st quadrant. Let's denote the part D:

$$D = \{0 \le \theta \le \pi/4, 4\sin\theta \le r \le 2\sqrt{2}\}.$$

$$\int_0^{\pi/4} \int_{4\sin\theta}^{2\sqrt{2}} rdrd\theta = \int_0^{\pi/4} [r^2/2]_{4\sin\theta}^{2\sqrt{2}} d\theta = \int_0^{\pi/4} 4 - 8\sin^2\theta d\theta = 4 \int_0^{\pi/4} 1 - 1 + \cos 2\theta d\theta$$

$$= 4[\frac{1}{2} \sin 2\theta]_0^{\pi/4} = 2[(\sin \pi/2) - (\sin 0)] = 2.$$

Thus the area we must find is $4 + 4\pi$.

(4) $x^2 + y^2 = 2x$ is $r = 2\cos\theta$ in polar coordinate. So the region in cylindrical coordinate is

$$D = \{-\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta, 0 \le z \le r^2\}$$

We can divide D into half disks and compute it over one of them since the region is symmetric along x-axis.

$$2 \int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{r^2} rdzdrd\theta = 2 \int_0^{\pi/2} \int_0^{2\cos\theta} r(r^2 - 0) drd\theta = 2 \int_0^{\pi/2} [\frac{1}{4} r^4]_0^{2\cos\theta} d\theta$$

$$= 8 \int_0^{\pi/2} \cos^4\theta d\theta = 8 \int_0^{\pi/2} 3 + 4\cos 2\theta + \cos 4\theta d\theta = 8[8\theta + 2\sin 2\theta + \frac{1}{4} \sin 4\theta]_0^{\pi/2} = 3\pi/2$$

where we applied the double angle formula $2\cos^2 x = 1 + \cos 2x$ twice.

(5)

$$D = \{0 \le \theta \le 2\pi, 0 \le z \le 4, 0 \le r \le \sqrt{z}\}$$

$$\int_0^{2\pi} d\theta \int_0^z \int_0^{\sqrt{z}} r^2 drdz = 2\pi \int_0^z z^{3/2}/2dz = \frac{2\pi}{3}[\frac{2}{5}z^{5/2}]_0^4 = \frac{4\pi 4^{5/2}}{15} = \frac{128\pi}{15}.$$

(6) (a)

$$D = \{0 \le y \le \pi/3, 0 \le x \le y\} = \{0 \le x \le \pi/3, x \le y \le \pi/3\}.$$

$$\int_0^{\pi/3} \int_x^{\pi/3} \cos(x+y)dydx = \int_0^{\pi/3} [\sin(x+y)]_x^{\pi/3} = \int_0^{\pi/3} \sin(x+\pi/3) - \sin(2x)dx$$

$$= [-\cos(x+\pi/3) + \frac{1}{2}\cos(2x)]_0^{\pi/3} = 1/4.$$

(b) The area of D is $\pi^2/18$. So the average of f over D is $\frac{1/4}{\pi^2/18} = 9/2\pi^2$. The mean value theorem says there is (x_0, y_0) in D such that $f(x_0, y_0)$ is the average. So there is (x_0, y_0) in D such that $\cos(x_0, y_0) = 9/2\pi^2$.