SPRING 2005 MATH 2220 SECOND PRELIMINARY PRACTICE EXAM

1. Let $f(x,y) = y^2 e^{y^2}$ and let D be the region in the xy-plane which is bounded by the lines $y = x/2$, $y = x$ and $y = a$ (where $a > 0$).

a) Set up two iterated integrals for $\int\int_D f(x,y)\, dA$.
b) Evaluate one of the integrals from part a).

a)

$$\int_0^a \int_y^{2y} y^2 e^{y^2}\, dx\, dy, \qquad \int_0^a \int_{x/2}^{x} xy^2 e^{y^2}\, dx\, dy + \int_a^{2a} \int_{x/2}^{a} y^2 e^{y^2}\, dx\, dy.$$

b)

$$\begin{aligned} \int_0^a \int_y^{2y} y^2 e^{y^2}\, dx\, dy &= \int_0^a y^3 e^{y^2}\, dy \\ &= \frac{1}{2}\int_0^{a^2} ue^u\, du = \frac{1}{2}[a^2 e^{a^2} - e^{a^2} + 1], \end{aligned}$$

where we have used the substitution $u = y^2$ and $du = 2y\, dy$.

2. Let $f(x,y,z) = x^2 - yz + z^3$.

a) Find the direction of fastest increase for the function f at the point $P = (1,2,1)$.
b) Find the equation of the tangent plane of the level surface

$$S = \{(x,y,z) \,|\, f(x,y,z) = 0\}$$

at the point $P = (1,2,1)$

a) Since

$$\nabla f(1,2,1) = (2x, -z, -y + 3z^2)|_{(1,2,1)} = (2,-1,1),$$

we conclude that the direction of fastest increase is given by the vector

$$\frac{\nabla f(1,2,1)}{\|\nabla f(1,2,1)\|} = \frac{(2,-1,1)}{\sqrt{6}}.$$

b) Since the vector $\nabla f(1,2,1) = (2,-1,1)$ is normal to the tangent plane, and the point $P = (1,2,1)$ belongs to the plane, we get that the required equation is

$$2x - y + z = 2(1) + (-1)(2) + 1(1) = 1.$$

3. Let

$$D = \{(x,y) \,|\, 4x^2 + y^2 \leq 1\}$$

and let $f(x,y) = x^2 + 4y^2$.

a) Find the critical points of f in the interior of D and classify them as local minimum, local maximum or saddle points.
b) Find the maximum and minimum value of f in the region D.

a) Observe that the equation

$$\nabla f(x,y) = (2x, 8y) = 0$$

has solutions only when $x = y = 0$. This point belongs to the interior of the region D. Now the matrix of second partial derivatives of f is given by

$$\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

Using the second derivative test at the critical points, we find that f has a local minimum at $(0,0)$.

b) To find the critical points of f on the boundary of D we will use the Lagrange multipliers method. That is, if we set $g(x,y) = 4x^2 + y^2 - 1$, we will look for solutions to the equations

$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y) \\ (2x, 8y) &= \lambda(8x, 2y). \end{aligned}$$

and

$$g(x,y) = 0.$$

The only solutions to this system of equations are

$$\lambda = 4, \qquad x = 0, \qquad y = \pm 1$$

and

$$\lambda = \frac{1}{4}, \qquad x = \pm\frac{1}{2}, \qquad y = 0.$$

Since

$$f(0,0) = 0, \qquad f(0,\pm 1) = 4, \qquad f(\pm\frac{1}{2}, 0) = \frac{1}{4},$$

we conclude that $(0,0)$ is the minimum point and both $(0,1)$ and $(0,-1)$ are the maximum points.

4. Consider the equation

$$ze^z - x^2 + xy - y^2 = 0.$$

a) Show that if (x_0, y_0, z_0) is a solution to this equation, then you can always find a local solution for z in terms of x and y.

b) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ implicitly.

a) We want to solve the equation $ze^z = x^2 - xy + y^2 = (x - y/2)^2 + 3y^2/4$ for z as a function of x and y. Since the right hand side is always non-negative and e^z is always positive, we conclude that any solution to this equations should satisfy $z \geq 0$. Now let

$$F(x,y,z) = ze^z - x^2 + xy - y^2.$$

Then $\partial F/\partial z = e^z + ze^z$. Since $z \geq 0$ everytime (x,y,z) is a solution to $F(x,y,z) = 0$, we conclude that $\partial F(x,y,z)/\partial z > 0$ for any such solution. Hence, by the implicit function theorem, we can always solve for z in terms of x and y, at least locally.

b) In this case

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = \frac{y - 2x}{e^z + ze^z}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = \frac{x - 2y}{e^z + ze^z}.$$

5. Let $f(x,y) = x\cos y + e^x \sin y$. Estimate $f(0.1, 0.2)$ using the second degree Taylor approximation of f starting at $(0,0)$.

Observe that

$$\frac{\partial f}{\partial x}(x,y) = \cos y + e^x \sin y, \qquad \frac{\partial f}{\partial y}(x,y) = -x\sin y + e^x \cos y,$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^x \sin y, \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -\sin y + e^x \cos y, \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = -x\cos y - e^x \sin y.$$

Hence, the second order Taylor polynomial of f at $(0,0)$ is given by

$$\begin{aligned} T(h,k) &= f(0,0) + \frac{\partial f}{\partial x}(0,0)h + \frac{\partial f}{\partial y}(0,0)k + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)h^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)hk + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)k^2 \\ &= h + k + hk. \end{aligned}$$

Hence,

$$f(0.1, 0.2) \approx T(0.1, 0.2) = 0.1 + 0.2 + 0.02 = 0.32.$$