MATH 2220 PRELIMINARY EXAM 1
FEBRUARY 19TH, 2015

Name____

1. Find the volume of the parallelepiped generated by the vectors

$$\mathbf{v} = (1, -1, 1) \qquad \mathbf{w} = (1, 0, 3) \qquad \mathbf{u} = (3, -2, 3). \qquad \text{(20pts)}$$

The volume of this parallelepiped is given by

$$|(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}|.$$

Since

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k},$$

then

$$|(\mathbf{v} \times \mathbf{w}) \cdot u| = |(-3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k})| = |-9 + 4 + 3| = 2,$$

is the volume.

2. Consider the curve parametrized by the equation

$$\mathbf{r}(t) = (e^t, \cos t, \cos 2t).$$

Find the equation of the line tangent to this curve at the point $P = (1, 1, 1)$. (20pts)

Observe that

$$\mathbf{r}(t) = (e^t, \cos t, \cos 2t) = (1, 1, 1)$$

if and only $t = 0$. Hence, the equation of the tangent line at $P = (1, 1, 1)$ is given by

$$\mathbf{l}(s) = \mathbf{r}(0) + s\mathbf{r}'(0).$$

Since $\mathbf{r}'(t) = (e^t, -\sin t, -2\sin 2t)$, we have that $\mathbf{r}'(0) = (1, 0, 0)$. Hence,

$$\mathbf{l}(s) = (1, 1, 1) + s(1, 0, 0) = (1 + s, 1, 1).$$

3. Find the equation of the plane containing the lines

$$\mathbf{r}(t) = (1,1,1) + t(1,-1,0)$$

and

$$\mathbf{s}(t) = (4,-1,0) + t(1,1,-2). \qquad \text{(20pts)}$$

First, let's observe that the equation

$$(1+t, 1-t, 1) = (4+s, -1+s, -2s)$$

has a unique solution when $t = 5/2$, $s = -1/2$. In other words, the lines corresponding to $\mathbf{r}$ and $\mathbf{s}$ intersect at the point $P = (7/2, -3/2, 1)$ and hence there exists a plane $\mathcal{P}$ containing this two lines. Now let

$$\mathbf{n} = (1,-1,0) \times (1,1,-2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

Since $\mathbf{n}$ is perpendicular to the lines $\mathbf{r}$ and $\mathbf{s}$, it should be normal to my plane $\mathcal{P}$. Since the point $(1,1,1)$ belongs to $\mathcal{P}$ (because it belongs to the line $\mathbf{r}$) an equation for this plane should be given by

$$(2,2,2) \cdot (x-1, y-1, z-1) = 0$$

or

$$2x + 2y + 2z = 6.$$

4. Let

$$f(x,y) = \begin{cases} \frac{4x^3-2xy^2+4y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Is f continuous at $(0,0)$? (20pts)

Using polar coordinates, we see that

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0} \frac{4r^3\cos^3\theta - 2r^3\cos\theta\sin^2\theta + 4r^3\sin^3\theta}{r^2} \\ &= \lim_{r\to 0} 4r\cos^3\theta - 2r\cos\theta\sin^2\theta + 4r\sin^3\theta = 0 = f(0,0). \end{aligned}$$

Hence, f is continuous at $(0,0)$.

5. Let

$$f(x, y) = x^2 - y^2.$$

i) Describe the level curves of f. (5pts)

ii) Let

$$\mathbf{r}(t) = (\cosh t, \sinh t).$$

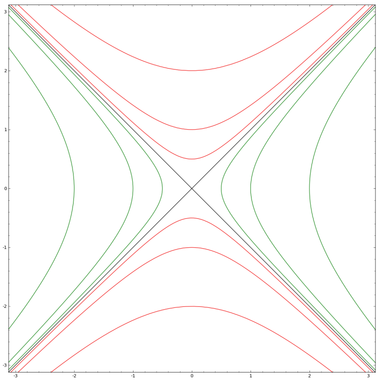
Show that the curve defined by $\mathbf{r}$ is contained in the level curve $f(x, y) = 1$. (5pts)

iii) Compute

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t), \qquad \text{for } t \in \mathbb{R}.$$

(Recall, $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$) (10pts)

i) Consider the level curve $f(x, y) = x^2 - y^2 = c$. If $c = 0$, then we obtain the lines $y = x, y = -x$. If $c > 0$, then we obtain hyperbolas that open along the x-axis starting at $\sqrt{c}$ and if $c < 0$ we obtain hyperbolas opening along the y-axis starting at $\sqrt{-c}$.



ii) We want to show that

$$f(r(t)) = f(\cosh t, \sinh t) = \cosh^2 t - \sinh^2 t = 1 \qquad \text{for all } t.$$

Using the definition

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2},$$

we have that

$$\cosh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4}, \qquad \sinh^2 t = \frac{e^{2t} - 2 + e^{-2t}}{4}.$$

Hence,

$$f(r(t)) = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = \frac{4}{4} = 1,$$

as we wanted to show.

iii) Since $\nabla f(x, y) = (2x, -2y)$, we have that

$$\nabla f(r(t)) = \nabla f(\cosh t, \sinh t) = (2\cosh t, -2\sinh t).$$

On the other hand, it is straightforward to check that

$$r'(t) = (\sinh t, \cosh t).$$

Hence

$$\begin{aligned} \nabla f(r(t)) \cdot r'(t) &= (\cosh t, -2\sinh t) \cdot (\sinh t, \cosh t) \\ &= 2\cosh t \sinh t - 2\sinh t \cosh t = 0, \end{aligned}$$

for all $t \in \mathbb{R}$.