MATH 2220 PRELIMINARY EXAM 2
MARCH 24TH, 2015

Name____

1. Let $F(x, y, z) = (x-1)^2 + y^2 + z^2/4 - 1$ and let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}.$$

Find all the points $P \in S$ such that the tangent plane to S at P is parallel to the plane

$$x + y + z = 1. \qquad \text{(15pts)}$$

We want to find all the points x, y, z such that

$$\nabla F(x, y, z) = (2(x-1), 2y, z/2) = \lambda(1, 1, 1) \qquad \text{and} \qquad F(x, y, z) = 0,$$

for some $\lambda \in \mathbb{R}$. Solving the first of this equations, we get that

$$x = \frac{\lambda}{2} + 1, \qquad y = \frac{\lambda}{2} \qquad z = 2\lambda.$$

Plugging this into the equation $F(x, y, z) = 0$, we get

$$1 = (\frac{\lambda}{2})^2 + (\frac{\lambda}{2})^2 + (4\frac{\lambda}{4})^2 = \frac{3}{2}\lambda^2$$

or $\lambda = \pm 2\sqrt{3}/3$. From all this we conclude that

$$(\frac{3+\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}) \qquad \text{and} \qquad (\frac{3-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}, \frac{-4\sqrt{3}}{3})$$

are the points we are looking for.

2. Let $f(x,y) = \sin(x+y)$.

a) Estimate $f(0.1, 0.1)$ using linear approximation. (5pts)

b) Show that the error term in the linear approximation above is less than 0.02. (10pts)

c) Can you show that the error term in the linear approximation given above is actually less than 0.002? (5pts)

a) Observe that

$$\frac{\partial f}{\partial x}(x,y) = \cos(x+y), \qquad \frac{\partial f}{\partial y}(x,y) = \cos(x+y).$$

Hence, the linear approximation is given by

$$f(0.1, 0.1) \cong f(0,0) + \frac{\partial f}{\partial x}(0,0)(0.1) + \frac{\partial f}{\partial y}(0,0)(0.1) = 0.2$$

b) We have that

$$R^1_{(0,0)}(0.1, 0.1) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(c_{1,1})(0.1)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}(c_{1,2})(0.1)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial x}(c_{2,1})(0.1)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(c_{2,2})(0.1)^2$$

for some points $c_{i,j}$ between $(0,0)$ and $(0.1, 0.1)$. Since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin(x+y),$$

we have that

$$|\frac{\partial^2 f}{\partial x^2}(x,y)| \leq 1$$

for all (x,y). Now, since $c_{1,1}$ lies between $(0,0)$ and $(0.1, 0.1)$ the above inequality is actually strict. Similarly

$$|\frac{\partial^2 f}{\partial x \partial y}(c_{1,2})| < 1 \qquad |\frac{\partial^2 f}{\partial y \partial x}(c_{2,1})| < 1 \qquad |\frac{\partial^2 f}{\partial y^2}(c_{2,2})| < 1.$$

Hence,

$$|R^1_{(0,0)}(0.1, 0.1) < \frac{1}{2}(4(0.1)^2) = 0.02$$

c) Since

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin(x+y),$$

we have that

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0.$$

Similarly,

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y^2}(0,0) = 0.$$

Hence, the second degree Taylor approximation of f at $(0,0)$ is the same as the linear approximation given above. But then

$$(1) \qquad R^2_{(0,0)} = \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(c_{1,1,1})(0.1)^3 + \cdots + \frac{1}{3!}\frac{\partial^3 f}{\partial y^3}(c_{2,2,2})(0.1)^3$$

for some points $c_{i,j,k}$ between $(0,0)$ and $(0.1, 0.1)$. Now

$$\frac{\partial^3 f}{\partial x^3}(x,y) = -\cos(x+y)$$

hence

$$|\frac{\partial^3 f}{\partial x^3}(c_{1,1,1})| \leq 1.$$

and similarly for the rest of the terms appearing on the right hand side of equation (1). Since there are eight such terms, we conclude that

$$|R^2_{(0,0)}| \leq \frac{8}{3!}(0.1)^3 < 2(0.001) = (0.002).$$

3. Let

$$D = \{(x, y) \,|\, x^2 + (y-1)^2 \le 4\}$$

and let $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = 3x - x^3 + y^2 - 2y.$$

Find the maximum and minimum of f on D. (20pts)

We will first look for critical points in the interior of D. Setting

$$\nabla f(x, y) = (3 - 3x^2, 2y - 2) = (0, 0),$$

we obtain the critical points

$$P_1 = (1, 1) \qquad \text{and} \qquad P_2 = (-1, 1).$$

Now we look for critical points on the boundary of D. Observe that in ∂D we should have $x^2 + (y-1)^2 = 4$, or equivalently,

$$y^2 - 2y = 3 - x^2.$$

Plugging this relationship back into f, we get that

$$f(x, y(x)) = 3x - x^3 + 3 - x^2.$$

In other words, finding the extreme points of f on ∂D is the same as finding the extreme points of the function

$$F(x) = 3x - x^3 + 3 - x^2 \qquad -2 \le x \le 2.$$

By the first deivative test, the critical points of F on $[-2, 2]$ are $x = -2$, $x = 2$ and

$$0 = F'(x) = 3 - 3x^2 - 2x \qquad \Longrightarrow \qquad x = \frac{-2 \pm \sqrt{40}}{6}.$$

Now since F is a cubic polynomial with negative leading coefficient, we know that F is decreasing between -2 and $\frac{-2-\sqrt{40}}{6}$, increasing between $\frac{-2-\sqrt{40}}{6}$ and $\frac{-2+\sqrt{40}}{6}$ and decreasing again between $\frac{-2+\sqrt{40}}{6}$ and 2. Evaluating, we have that

$$F(-2) = -6 + 8 + 3 - 4 = 1$$

and

$$F(2) = 6 - 8 + 3 - 4 = -3.$$

To evaluate F on the other two critical points we observe that since $6 \le \sqrt{40} \le 7$ we should have that

$$\frac{2}{3} \le \frac{-2 + \sqrt{40}}{6} \le \frac{5}{6}$$

and

$$-\frac{3}{2} \le \frac{-2 - \sqrt{40}}{6} \le -\frac{4}{3}.$$

Therefore,

$$\begin{aligned} F\left(\frac{-2+\sqrt{40}}{6}\right) &\ge 2 - \left(\frac{5}{6}\right)^3 + 3 - \left(\frac{5}{6}\right)^2 \\ &= 5 - \frac{25}{36}\frac{11}{6} > 3. \end{aligned}$$

and

$$\begin{aligned} F\left(\frac{-2-\sqrt{40}}{6}\right) &\geq -3\left(\frac{3}{2}\right)+\left(\frac{4}{3}\right)^3+3-\left(\frac{3}{2}\right)^2 \\ &= -1+\frac{10}{27}-\frac{3}{4} > -2 \end{aligned}$$

From all of this we conclude that

$$F\left(\frac{-2+\sqrt{40}}{6}\right) \geq 3 \geq F(-2) = 1 \geq F\left(\frac{-2-\sqrt{40}}{6}\right) \geq -2 \geq F(2) = -3.$$

Finally, we observe that

$$f(P_1) = f(1,1) = 1 \qquad \text{and} f(P_2) = f(-1,1) = -3.$$

Therefore we conclude that the absolute minimal points are $(-1,1)$ and $(2,0)$ and the absolute maximal point occurs when $x = \frac{-2+\sqrt{40}}{6}$ and $x^2 + (y-1)^2 = 4$.

4. Let

$$F_1(x,y,u,v) = e^x y^2 + v\sin(x+u)$$

and

$$F_2(x,y,u,v) = x + y + v\cos(x+u).$$

Show that the equations $F_1 = 0 = F_2$ can be used to find u and v in terms of x and y in a neighborhood of $(-1,0,1,1)$. (15pts)

Observe that

$$\tfrac{\partial F_1}{\partial u} = v\cos(u+x) \qquad \tfrac{\partial F_1}{\partial v} = \sin(u+x)$$

$$\tfrac{\partial F_2}{\partial u} = -v\sin(u+x) \quad \tfrac{\partial F_2}{\partial v} = \cos(u+x).$$

Hence,

$$\left.\frac{\partial(F_1,F_2)}{\partial(u,v)}\right|_{(-1,0,1,1)} = (v\cos^2(u+x) + v\sin^2(u+x))|_{(-1,0,1,1)} = v|_{(-1,0,1,1)} = 1 \neq 0.$$

Therefore, from the implicit function theorem, we can solve for u and v in terms of x and y in a neighborhood of $(-1,0,1,1)$.

5. Let $\mathbf{c}(t) = (t, t^2, 2t^3/3)$ for $t \in [0, 1]$. Compute the arc length of the curve defined by the path $\mathbf{c}$. (15pts)

Since

$$\mathbf{c}'(t) = (1, 2t, 2t^2),$$

we have that

$$\|\mathbf{c}'(t)\| = (1 + 4t^2 + 4t^2)^{\frac{1}{2}} = [(1 + 2t^2)^2]^{\frac{1}{2}} = 1 + 2t^2.$$

Hence

$$\int_0^1 \|\mathbf{c}'(t)\| \, dt = \int_0^1 (1 + 2t^2) \, dt = 1 + \frac{2}{3} = \frac{5}{3}.$$

6. Let D be the region bounded by the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Compute

$$\iint_D \sqrt{1-x^2}\,dA. \qquad \text{(15pts)}$$

Observe that

$$\begin{aligned}\int\int_D \sqrt{1-x^2}\,dy\,dx &= \int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx \\ &= \int_0^1 x\sqrt{1-x^2}\,dx \\ &= -\frac{(1-x^2)^{\frac{3}{2}}}{3}\Bigg|_{x=0}^{x=1} \\ &= \frac{1}{3}.\end{aligned}$$