1 The Probabilistic Method

Learning Goals: Students should be able to

- **explain** what is meant by "the probabilistic method" and why it works

The probabilistic method was pioneered by Paul Erdös (of the Erdös-Rényi random graph) as a tool in combinatorics, the mathematics of counting. The probabilistic method uses probability to prove the existence of a mathematical object that has a certain property.

The Probabilistic Method:

1.

2.

3.

Exercise 1. Prove that the probabilistic method is valid (i.e., if steps 1 and 2 hold, then the conclusion in step 3 follows).

The probabilistic method is useful in cases where it is difficult to explicitly construct an object with property A. The method is *non-constructive*, meaning that it does not tell you explicitly which object has the desired property, only that one must exist.

This doesn't make the method entirely impractical, though. If the probabilistic method tells you that an object with property A has high probability, then you have a good chance of constructing one by sampling one at random. This is often computationally easier than checking all the objects until you find one that has the desired property.

2 Tournaments

Learning Goals: Students should be able to

- **prove** the existence of mathematical objects using the probabilistic method,
- **derive** bounds on quantities using the probabilistic method.

We will use the probabilistic method to prove the existence of *tournaments* having a certain property. We will also find an upper bound on how small a tournament with this property can be.

Definition. A *tournament* on n vertices is a directed graph on n vertices such that:

A tournament can be interpreted as the outcome of a round-robin tournament in which every player faces every other player exactly once. Given two players x and y, we draw an arrow from x to y if x beats y (we say "x dominates y"), and we draw an arrow from y to x if y beats x (there are no ties).

A random tournament is one in which each directed edge has probability $\frac{1}{2}$ to point in either of the two possible directions, independently of all the other edges.

Exercise 2. Let P_1 be the property that every player is dominated by some other player. Give an explicit example to show that for $n \geq 3$, there is a tournament of size n with property P_1.

Exercise 3. Let P_2 be the property that for every pair of players, there is a third player who dominates both of them. Show that no tournament on 3 or 4 vertices satisfies property P_2.

Adding more vertices might make it easier to satisfy P_2, because there are more choices for who can dominate a given pair. However, every additional vertex also must also have all of its pairs dominated. This makes it difficult to explicitly draw a tournament satisfying property P_2. The probabilistic method can help.

Proposition 1. *There exists a tournament satisfying property P_2.*

Exercise 4. Using the proof of the previous proposition, give an explicit number n such that there exists a tournament on n vertices having property P_2. What is the smallest value of n you know will work? Is it possible a smaller value of n could work?

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2	(n choose 2)*(3/4)^(n-2)	4.22	4.75	4.98	4.98	4.81	4.51	4.13	3.72	3.29	2.88	2.49	2.14	1.82	1.53	1.29	1.07	0.89	0.73

Exercise 5. Find a value of n such that a random tournament on n vertices satisfies $\mathbb{P}(P_2) \geq 0.99$.