1 Review of the Galton-Watson Process

Recall the following definitions:

- **Offspring distribution**: $(p_0, p_1, \dots)$
  - p_k is interpreted as the probability that an individual has k offspring
- **Mean of the offspring distribution**: $\mu = \sum_k k p_k$
  - μ is interpreted as the average number of offspring of any individual
- $\xi_{n,i}$: the number of offspring of the ith individual in the nth generation (sampled independently from the offspring distribution)
- Z_n: the size of the nth generation
  - $Z_0 = 1$
  - $Z_{n+1} = \xi_{n,1} + \xi_{n,2} + \cdots + \xi_{n,Z_n}$

Last time we proved:

$$\mathbb{E}[Z_n] = \mu \mathbb{E}[Z_{n-1}], \text{ so } \mathbb{E}[Z_n] = \mu^n.$$

The first question we will study (and the first question Francis Galton posed) is:

What is the probability of extinction (i.e., eventually some generation has size zero)?

2 Probability of Extinction: $\mu < 1$

In this section we assume $\mu < 1$. In this case, individuals are not having enough offspring on average to sustain the population. The average size of the n generation, μ^n, is decreasing exponentially with n. This suggests that the family should go extinct with probability 1. We can prove this in two ways.

Exercise 1. Prove that $\mathbb{P}(Z_n = 0) \geq 1 - \mu^n$. Use this to show that the probability of extinction is 1, i.e.,

$$\mathbb{P}(Z_n = 0 \text{ for some } n \geq 0) = 1.$$

Explain why this proof doesn't work when $\mu \geq 1$.

We can also prove this by looking at the expected size of the entire family.

Definition. T_n denotes the size of the total family up to the nth generation, i.e.,

$$T_n = Z_0 + Z_1 + \cdots + Z_n.$$

T denotes the size of the entire family, i.e.,

$$T = \lim_{n\to\infty} T_n = Z_0 + Z_1 + Z_2 + \cdots$$

which might be finite or infinite.

Exercise 2. Let X be a random variable taking values in $\{0, 1, 2, \dots\} \cup \{\infty\}$. Which of the following statements are true?

- If $\mathbb{E}[X] < \infty$, then $\mathbb{P}(X < \infty) = 1$.
- If $\mathbb{P}(X < \infty) = 1$, then $\mathbb{E}[X] < \infty$.

Exercise 3. Prove that if $\mathbb{E}[T] < \infty$, then $\mathbb{P}(Z_n = 0 \text{ for some } n \geq 0) = 1$.

Exercise 4. Calculate $\mathbb{E}[T]$ when $\mu < 1$. Conclude that the probability of extinction is 1 in this case. Explain why this doesn't work when $\mu \geq 1$.

3 Probability of Extinction: $\mu \geq 1$

Based on the calculations for $\mu < 1$, we may wonder what conclusions we can make for $\mu \geq 1$.

Exercise 5. Can we conclude that $\mathbb{P}(Z_n = 0 \text{ for some } n \geq 0) < 1$ in the case $\mu \geq 1$?

We need another way to determine the extinction probability when $\mu \geq 1$. Let the extinction probability be $q = \mathbb{P}(Z_n = 0 \text{ for some } n \geq 0)$.

Exercise 6. Use the law of total probability and conditioning on the value of Z_1 to show that q satisfies the equation:

$$q = p_0 + p_1 q + p_2 q^2 + p_3 q^3 + \cdots .$$

Does this look familiar?

4 A Note on Terminology

Why do I say "go extinct with probability 1" instead of "go extinct"? What is the difference between something happening with probability 1 and something happening for sure? Some examples can illuminate the difference.

Exercise 7. Let X be a sampled uniformly from the interval $[0, 1]$. What is the probability that $X \neq \frac{1}{2}$? Can you be sure that $X \neq \frac{1}{2}$?

Exercise 8. Consider flipping an infinite sequence of coins. What is the probability that you do not get all heads? Can you be sure you don't get all heads?

Definition. We say an event A happens **almost surely** if it happens with probability 1, i.e., $\mathbb{P}(A) = 1$. We say an event A happens **surely** if it is equal to the entire sample space.

An almost sure event A does not have to equal the entire sample space, but rather the sample space minus some probability-zero event (i.e., an event E with $\mathbb{P}(E) = 0$). In the first example, the event $X \neq \frac{1}{2}$ is all of the sample space except the probability-zero event that $X = \frac{1}{2}$. In the second example, the event that you don't flip all heads is the whole sample space, minus the probability-zero event that you do flip all heads.

We proved that the Galton-Watson branching process goes extinct almost surely if $\mu < 1$. In other words, with probability 1 the family tree is finite. However, if it is possible for every individual to have a positive number of offspring, then it is possible for the family tree to be infinite. Hence the event of extinction is almost sure, but not sure.

The flip side of this is that something happening with probability zero is not the same as it being impossible! Probability zero events are possible, and they happen every day!

References

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