1 Branching Process Review

Recall the following definitions:

- **Offspring distribution**: $(p_0, p_1, \ldots)$
  - p_k is interpreted as the probability that an individual has k offspring
- **Mean of the offspring distribution**: $\mu = \sum_k kp_k$
  - μ is interpreted as the average number of offspring of any individual
- $\xi_{n,i}$: the number of offspring of the ith individual in the nth generation (sampled independently from the offspring distribution)
- Z_n: the size of the nth generation
  - $Z_0 = 1$
  - $Z_{n+1} = \xi_{n,1} + \xi_{n,2} + \cdots + \xi_{n,Z_n}$
- **The probability generating function of the offspring distribution**:

$$f(x) = \sum_{k=0}^{\infty} p_k x^k$$

- **The probability generating function of Z_n**:

$$f_n(x) = \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)x^k$$

- **The probability of extinction**:

$$q = \mathbb{P}(\exists\ n \text{ s.t. } Z_n = 0) = \lim_{n\to\infty} \mathbb{P}(Z_n = 0)$$

We have proved the following results:

- $\mathbb{E}[Z_n] = \mu^n$
- $f_n(x) = \underbrace{f(f(\cdots f}_{n}(x)\cdots)) = f^n(x)$
- If $\mu < 1$ then $q = 1$

Learning Goals: Students should be able to

- **calculate and interpret** quantities involving probability generating functions
- **sketch** the probability generating function of the offspring distribution
- **calculate** the probability of extinction of the branching process

2 Putting Generating Functions to Use

The generating function of a random variable encodes its entire distribution in one function. Therefore, we can study the distributions of random variables by manipulating their generating functions.

Recall that for any random variable X, we calculated that its generating function $f_X(x)$ satisfies:

$$f_X(1) = 1, \qquad f_X'(1) = \mathbb{E}[X].$$

Thus $f_n(x)$, the generating function of Z_n, satisfies:

$$f_n(1) = 1, \qquad f_n'(1) = \mathbb{E}[Z_n].$$

Let μ denote the mean of the offspring distribution, i.e., $f'(1)$.

Exercise 1. Use the chain rule and the fact that $f_2(x) = f(f(x))$ to calculate $\mathbb{E}[Z_2]$.

Exercise 2. Use the chain rule and the fact that $f_n(x) = f_{n-1}(f(x))$ to provide an alternate proof that $\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}]$. Conclude that $\mathbb{E}[Z_n] = \mu^n$ for all n.

Exercise 3. If $f_n(x)$ is the p.g.f. of Z_n, what is the interpretation of $f_n(0)$? $\lim_{n\to\infty} f_n(0)$?

3 Some Assumptions

We will make a few simplifying assumptions about our offspring distribution $(p_0, p_1, p_2, \dots)$.

1. Assume $p_k \neq 1$ for any k.

2. Assume $p_0 + p_1 < 1$. That is, there exists $k \geq 2$ such that $p_k > 0$.

The first assumption just says that individuals don't always have exactly k offspring. If we had $p_k = 1$ for all k, then the process is not random at all and we just have $Z_n = k^n$ for all n.

The second assumption says that it is possible for an individual to have at least two offspring. If we had $p_0 + p_1 = 1$, then every offspring has exactly zero or one offspring.

Exercise 4. Suppose $p_0 + p_1 = 1$ (and neither p_0 nor p_1 is equal to 1). Argue that

$$\mathbb{P}(Z_n = 1) = (p_1)^n.$$

What is the extinction probability in this case?

Exercise 5. Consider any offspring distribution $(p_0, p_1, p_2, \dots)$ that satisfies the assumptions above.

- Is $f(x)$ increasing or decreasing on $(0, 1)$?
- What is the concavity of $f(x)$ on $(0, 1)$?

4 Probability of Extinction, $\mu \geq 1$

We saw in Exercise 3 that $f_n(0) = \mathbb{P}(Z_n = 0)$, while the probability of extinction is:

$$q = \mathbb{P}(\exists\ n \text{ s.t. } Z_n = 0) = \lim_{n\to\infty} \mathbb{P}(Z_n = 0).$$

Therefore, the probability of extinction is:

$$q = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} f^n(0).$$

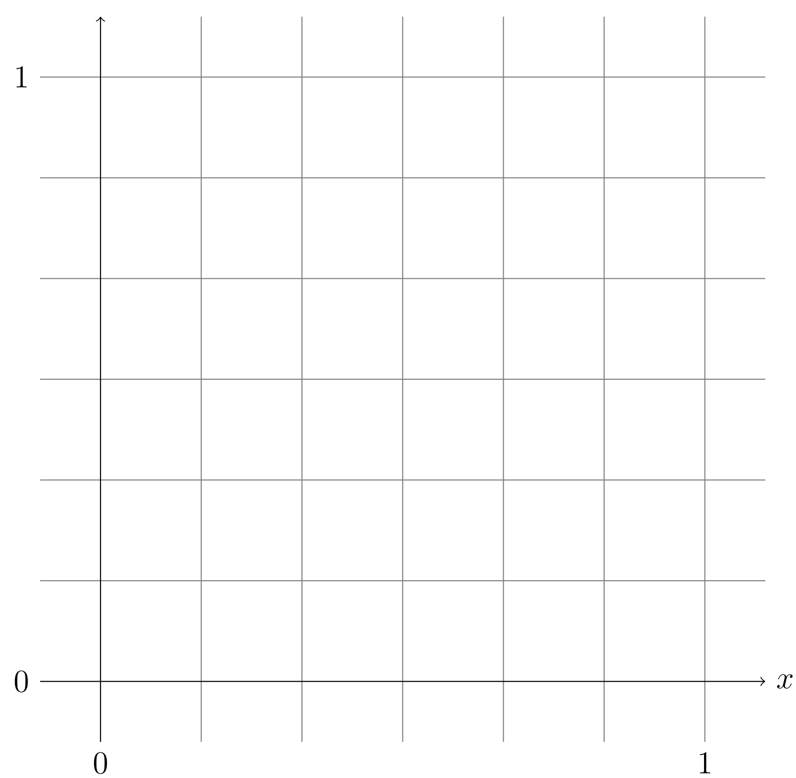
So we need to understand what happens when we **iterate** $f(x)$ starting at $x = 0$.

4.1 Example 1: $\mu > 1$

Consider the offspring distribution with $p_0 = \frac{1}{3}$ and $p_2 = \frac{2}{3}$.

Exercise 6. What is $f(x)$ (the p.g.f. of the offspring distribution) in this case? Calculate $f(0)$, $f(1)$ and $f'(1)$. What is the mean μ of the offspring distribution?

Exercise 7. Using your answers to Exercises 5 and 6, sketch $f(x)$ on the interval $[0, 1]$ on the axes below.



Exercise 8. Add a sketch of the function $y = x$ to the graph above. What do you know about the slope of this function, compared to the slope of $f(x)$, at $x = 1$? Adjust your sketch of $f(x)$ if necessary to reflect this information.

Exercise 9. Sketch the point $(0, f(0))$ on the graph. Draw a horizontal line from this point until you hit the line $y = x$. What is the x-coordinate of this intersection point (in terms of $f(x)$)?

Exercise 10. From the intersection point in the previous exercise, draw a vertical line until you hit the curve $y = f(x)$. What are the coordinates of this intersection point (in terms of $f(x)$)?

Exercise 11. Starting from the last intersection point in the previous exercise, suppose you continue drawing a horizontal line until you hit $y = x$ and then a vertical line until you hit $y = f(x)$. (The resulting diagram is called a **cobweb plot**.) What will be the coordinates of the points on the line $y = f(x)$? Where will they be on the graph?

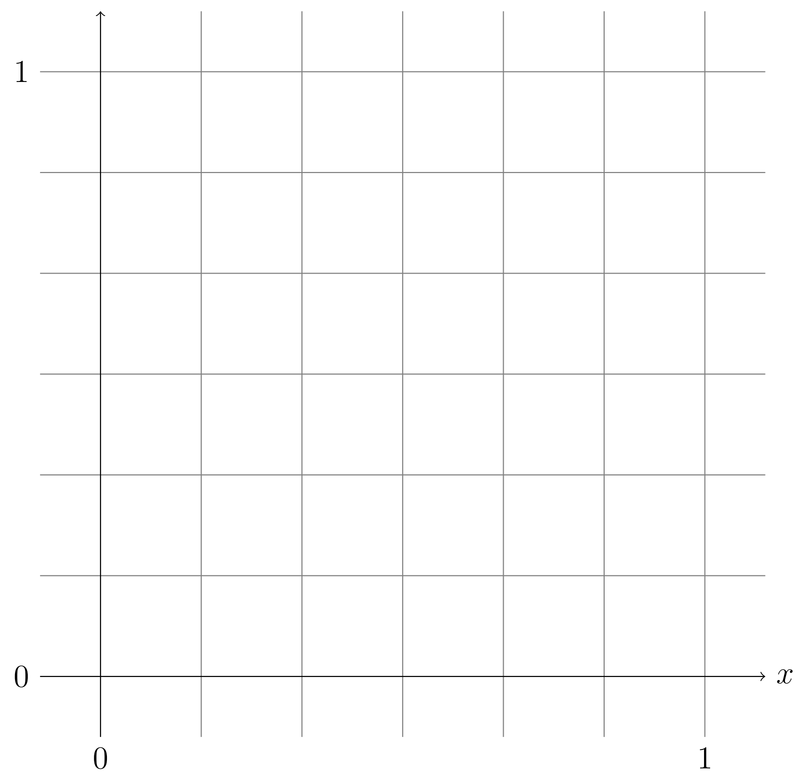
Exercise 12. What is the extinction probability q of this branching process in terms of the graphs $y = f(x)$ and $y = x$? Can you calculate its exact value?

4.2 Example 2: $\mu = 1$

Consider the offspring distribution with $p_0 = p_1 = p_2 = \frac{1}{3}$.

Exercise 13. What is $f(x)$ (the p.g.f. of the offspring distribution) in this case? Calculate $f(0)$, $f(1)$ and $f'(1)$. What is the mean μ of the offspring distribution?

Exercise 14. Using your answers to Exercises 5 and 13, sketch $f(x)$ on the interval $[0, 1]$ on the axes below.



Exercise 15. Add a sketch of the function $y = x$ to the graph above. What do you know about the slope of this function, compared to the slope of $f(x)$, at $x = 1$? Adjust your sketch of $f(x)$ if necessary to reflect this information.

Exercise 16. Sketch the cobweb diagram starting at the point $(0, f(0))$. Where will the intersection points be on the graph? What is the extinction probability q of this branching process in terms of the graphs $y = f(x)$ and $y = x$? Can you calculate its exact value?

4.3 Results

The exploration in the previous section leads us to the following theorem.

Theorem 1 (Probability of Extinction)**.** *Suppose* $(p_0, p_1, \dots)$ *is an offspring distribution satisfying the assumptions of Section 3. Let* $f(x)$ *be the p.g.f. of this offspring distribution. Then the extinction probability* q *of the branching process with this offspring distribution is the smallest nonnegative solution to the equation* $x = f(x)$*. It satisfies:*

- $q = 1$ *when* $\mu \leq 1$*,*
- $0 \leq q < 1$ *when* $\mu > 1$*.*

We have not rigorously proved this result, but it should be believable based on the examples above.

Exercise 17.

(a) What about Example 1 made it so that $q < 1$? That is, why did the graphs $y = x$ and $y = f(x)$ intersect somewhere with $x < 1$?

(b) What about Example 2 made it so that $q = 1$? That is, why did the graphs $y = x$ and $y = f(x)$ not intersect somewhere with $x < 1$?

(c) We did not draw a cobweb diagram for an example with $\mu < 1$, but suppose you did. How would this differ from Examples 1 and 2? What would q be in this case?

Our exploration with cobweb diagrams suggests that q satisfies $q = f(q)$, but we can actually prove this directly. Note, however, that there can be more than one solution to this equation, so we really need Theorem 1 to tell us which solution gives the probability of extinction.

Exercise 18. Use the law of total probability and conditioning on the value of Z_1 to show that q satisfies the equation:

$$q = p_0 + p_1 q + p_2 q^2 + p_3 q^3 + \cdots .$$

That is, $q = f(q)$.