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DIMENSION ESTIMATES FOR ATTRACTORS

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Numerical and theoretical studies of three dimensional flows and one and two dimensional iterations have yielded a coherent picture of "chaotic" dynamics in its simplest forms. This body of knowledge is relevant for the experimental and analytical study of fluid dynamics in regimes which represent the transition to "turbulence". Here turbulence is used loosely as referring to aperiodic flow with a continuous power spectrum. This paper is a discussion and review of aspects of dynamical systems theory which appear to be useful in the interpretation of experimental observations together with some new remarks about the statistical problems of estimating the Hausdorff dimensions of attractors. The methods I describe are of more general applicability than just to fluid experiments, but I have restricted myself to procedures which appear feasible with the amount of data which is readily available from work with fluids.

The issues which I address involve determining whether the state of a fluid can be represented by a "reduced" model with few degrees of freedom. If one assumes that the system is behaving in a deterministic fashion, then one would like a dimension estimate for the attractors which occur in the state space of the fluid. Once transients in the fluid system have decayed, the observed system follows a trajectory in state space whose closure is called an attractor of the system. For chaotic systems, attractors typically have a frightful topological structure which makes even the definition of dimension problematic. Given the lack of clear understanding of the finest details of the chaotic motion in simple models such as the forced Duffing equation or the Henon mapping [7], it is unreasonable that any statistical method can be proved to give accurate estimates of dimension for all attractors. The best that one can hope for is a procedure which is reliable for classes of well understood examples and that the scope of these classes of examples might be enlarged in the future. Here I concentrate on a simple, computationally inexpensive statistical method and discuss a few examples, each of whose analysis contains only some of

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the difficulties of the general problem. Hopefully, these heuristic considerations will be helpful in understanding the practical efforts to compute the dimension of attractors from experimental data.

1. DIMENSIONS AND ESTIMATES. Farmer, Ott and Yorke [3] have reviewed a number of related concepts of fractional dimensions and their application to attractors. The reader should consult this paper for additional discussion and background about dimensions.

Definitions of dimension which depend upon a specific measure rather than just a geometric set of points in state space are relevant for the study of attractors. The most reasonable view of this matter (based on experience with numerical simulations) appears to be the following. Initial conditions for a flow are interpreted as avoiding sets of zero Lebesgue measure with special properties. Properties which hold in sets of positive Lebesgue measure are observed in watching long trajectories. Different trajectories in a chaotic attractor may have different asymptotic properties, but I <u>assume</u> that there is a unique, ergodic probability measure upon the attractor Λ which represents the asymptotic properties of almost all trajectories in its basin of attraction. This means that

$$\lim \frac{1}{T} \int_0^T f(\varphi(x,t))\,dt = \int_\Lambda f\,d\mu$$

for almost all x in the basin of attraction of Λ and μ-integrable functions f. The existence and uniqueness of μ are proved for hyperbolic attractors by Sinai, Bowen, and Ruelle [2] and for a large class of one dimensional mappings by Jakobson [9]. Sets of μ-measure zero should be excluded from dimension calculations, even if they have larger Hausdorff dimension than a set of full μ-measure. This is evident in the generalized baker's transformation discussed below.

The choice of calculation strategy is an important one. Procedures based upon "box counting" techniques [5] require large amounts of computational time, and the computational time grows rapidly with the dimension of the set being described. The procedure which I analyze here is called the <u>pointwise dimension</u> by Farmer et. al.

For an attractor Λ, $x \in \Lambda$, and a probability measure μ supported on Λ, one defines

$$d_p(x) = \lim_{r \to 0} \frac{\log \mu(B(r,x))}{\log r}$$

where $B(r,x)$ is the ball of radius r centered at x in the state space of the flow. If $d_p(x)$ exists and is independent of x for μ-almost all $x \in \Lambda$, then this is defined to be the pointwise dimension of Λ. For simplicity, I describe the procedures in terms of discrete time. One may think in terms of a discrete sampling of a continuous system <u>or</u> in terms of a Poincaré section of a continuous system for the purposes of these measurements.

Estimation of $d_p(x)$ from a trajectory of length n can be performed efficiently. The technique is to first compute the distances $\delta(t) = |x - \phi_t(y)|$ from x to each point $\phi_t(y)$ in the trajectory based at y. The assumption of the existence of an asymptotic measure implies that if $N(r)$ is the number of $\delta(t)$ with $\delta(t) < r$, then $\frac{N(r)}{n}$ is a good approximation to $\mu(B(x,r))$ when n is large. The numbers $N(r)$ are easily calculated by sorting the n numbers $\delta(t)$. One can then plot $\log N(r)$ versus $\log r$ to estimate the limit value of $\log \mu(B(x,r))/\log r$ as $r \to 0$. The number of operations needed to calculate the $\delta(t)$ grows like $n \cdot k$, k being the dimension of phase space. The number of operations needed to sort the $\delta(t)$ is of order $n \log n$. Thus the computational time has a total order of magnitude which is $n(k + \log n)$. This is significantly smaller than the computational time required by other methods [4] and is the only dimension estimate which appears feasible on a minicomputer at this time. The practical questions in the implementation of the method involve (1) the size of the statistical fluctuations which one should expect as one varies n and (2) the optimal way to extrapolate to the limit $r \to 0$. The first of these issues is considered here in terms of examples, following a review of some of the properties of <u>order statistics</u>.

2. ORDER STATISTICS [W]. If $X_i, i=1,\cdots,n$ are independent random variables with the same continuous cumulative distribution function $F(x)$, then almost surely $X_i \neq X_j$ for $i \neq j$. If the X_i's are ordered to yield $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ with $X_{(i)} \in \{X_1, \cdots, X_n\}$, then the $X_{(i)}$ are called the <u>order statistics</u> of the X_i. The order statistics of $\{F(X_i)\}$ are independent of the distribution function F since each $F(X_i)$ is a uniform random variable on the interval $[0,1]$. Therefore, it is possible to study the order statistics $F(X_{(i)})$ in a setting which is completely non-parametric (not dependent upon a distribution).

The theory of order statistics plays an important role in our estimates of dimension because the distances $\delta(t)$ are sorted (i.e., arranged in numerical order) as part of the computational process. One may view the $\delta(t)$ as random variables whose distribution is given by $V(r) = \mu(B(r,x))$. In other words, the probability that $\delta(t)$ is smaller than r is the μ-volume of the ball of radius r centered at x. The order statistics $\delta_{(i)}$ are used to estimate the distribution $V(r)$ by assuming that $V(\delta_{(i)}) \approx \frac{i}{n}$ (with n the number of computed distances). Built into this procedure is the implicit assumption that the $\delta(t)$ are independent of one another. If the sampling interval is sufficiently long, then sensitivity to initial conditions suggests that this is a reasonable condition for chaotic attractors*. The theory of order statistics then describes the distribution of the quantities $v_{(i)} = V(\delta_{(i)})$ for varying realizations of the random variables $\phi_t(y)$.

I do not go into much detail concerning order statistics here, but note that the

*Some continuous-time attractors appear to have a long time "phase coherence" that must be accounted for here. For quasiperiodic attractors, the $\phi_t(y)$ will not be independent, but they will typically be uniformly distributed.

distribution of the i^{th} observed volume $v_{(i)}$ is given by the Beta distribution $Be(i,n-i+1) = \frac{n!}{(i-1)!(n-i)!} v^{i-1}(1-v)^{n-i}$ which has mean $\frac{i}{n+1}$ and variance $\frac{i(n-i+1)}{(n+1)^2(n+2)}$. If $i = k - j$, then $Be(i,n-i+1)$ also gives the distribution of $v_{(k)} - v_{(j)}$. Wilks [10] also includes information concerning the joint distribution of different order statistics. As an illustration of these ideas, I discuss the estimation of the dimension of a torus and a cube of dimension d using these methods.

Consider a torus $T^d = S^1 \times \cdots \times S^1$ with distance function $\delta(\theta,\psi) = \max_{1 \le j \le d} \delta(\theta_j - \psi_j)$. Here $\delta(\theta_j,\psi_j)$ is the length of the shortest arc on S^1 joining θ_j and ψ_j: $\delta(\theta_j,\psi_j) = \min(|\theta_j - \psi_j|, \; 2\pi - (\theta_j - \psi_j)|)$ if $0 \le \theta_j, \psi_j < 2\pi$. The simplicity of this example for dimension estimates lies in the fact that the open ball of radius π centered at $\underline{\theta} \in T^d$ covers almost all of T^d and the volume of the ball of radius r, $0 < r < \pi$ is $(\frac{r}{\pi})^d$ (with the volume of T^d normalized to be 1). Thus the function logV vs. logr to be estimated is a straight line of slope d. From a random sample of n points ψ_i on T^d, an estimate for d can be obtained from the order statistics $\delta_{(i)}$ of $\delta_i = \delta(\theta,\psi_i)$. One need only pick two fixed values of $\frac{k}{n}$, $\frac{\ell}{n}$ and estimate $\frac{\log(k) - \log(\ell)}{\log \delta_{(k)} - \log \delta_{(\ell)}}$ as the value of d. The variance in the estimates of V will be order $\frac{1}{\sqrt{n}}$. This implies that, for fixed $\frac{k}{n}$ and $\frac{\ell}{n}$ and n large, the variance in the estimate of d will be of order $\frac{d}{\sqrt{n}}$. A more complete statistical analysis of the variance in the estimate for d is possible. Note, however, that the relative precision of this estimate is independent of d. Figure 1 illustrates the results of a numerical computation of logV vs. logr for $n = 5000$ and $d = 2, 3, 5, 10, 25, 50,$ and 99.

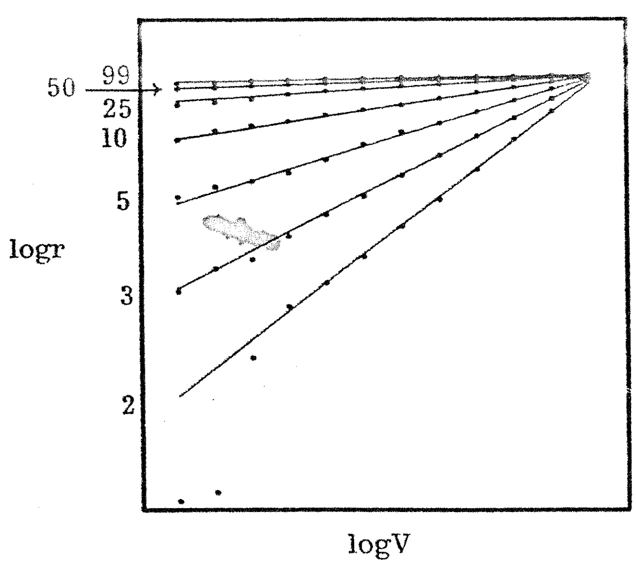

Figure 1

As a second illustration of these dimension estimates, I consider an example in which the estimates depend upon the choice of reference point x. Denote the unit cube $I^d = ([0,1])^d$ in d-dimensional Euclidean space with the sup norm $\delta(\underline{x},\underline{y}) = \max_{1 \le j \le d} (|x_j - y_j|)$. For the reference point x chosen as the center or one vertex of the cube, the analysis of this example is essentially the same as the toral example discussed above. However, if x is chosen randomly in I^d, then there are new complications in the dimension estimates due to the fact that x will be closer to faces of the cube normal to some coordinate axes than to others. In particular if $e_{(1)}, \cdots, e_{(d)}$ are the order statistics of $e_i = \min(x_i, 1-x_i)$, then one can explicitly compute that $V(r) = \epsilon_1 \cdots \epsilon_d$ where

$$\epsilon_j = \begin{cases} 2r & \text{if } r \le e_{(j)} \\ r + e_{(j)} & \text{if } e_{(j)} \le r \le 1 - e_{(j)} \\ 1 & \text{if } 1 - e_{(j)} \le r \end{cases}$$

Consequently, $\log V$ will be a piecewise smooth function of $\log r$ which is linear with slope d only for $r < e_{(1)}$ (assuming $\underline{x}$ is in the interior of the cube). In this situation, estimating dimension from the order statistics $\delta_{(i)}$ leads to an under-estimate of d due to the presence of boundaries of I^d. The geometric structure of the support of the measure whose dimension one is trying to estimate can bias these procedures. To obtain accurate estimates of dimension, one needs information about the shortest length scales for which these geometric effects play a role in the dimension estimates. Similar effects can be seen in dimension estimates for a rectangular solid with different edge lengths but reference point in the center. If the edge lengths of a rectangular solid are $2e_1 < 2e_2 < \cdots < 2e_d$, then $\log V$ is a linear function of $\log r$ of slope $d-j$ in the range $e_j < r < e_{j+1}$.

3. CANTOR SETS, FRACTALS, AND ATTRACTORS. The final ingredient which affects the estimation of Hausdorff dimensions for attractors is their complicated "fractal" structure. This results in a situation for which the volume function $V_x(r)$ is likely to lose smoothness, even though it can be expected to vanish like r^d as $r \to 0$. I shall present a simple illustration of this phenomenon and then discuss two examples.

Consider the standard Cantor set $C \subset [0,1]$ defined by $C = \{\sum_{i=1}^{\infty} a_i 3^{-i} \mid a_i = 0 \text{ or } 2 \text{ for all } i\}$. C supports an invariant probability measure μ for which each of the sets $C_{b_1,\cdots,b_k} = \{\sum_{i=1}^{\infty} a_i 3^{-i} \mid a_i = b_i \text{ for } 1 \le i \le k\}$ has volume $\frac{1}{2^n}$. The volume function $V_x(r)$ corresponding to μ is easily computed for arbitrary x; I take $x = 0$ for the sake of elegance. Explicitly, $V_0(r)$ is given by the Cantor function defined first on C by $V_0(\Sigma a_i 3^{-i}) = \Sigma \frac{a_i}{2} 2^{-i}$ and then by extending V_0 to be constant on each component of the complement of C. See Figure 2. The function

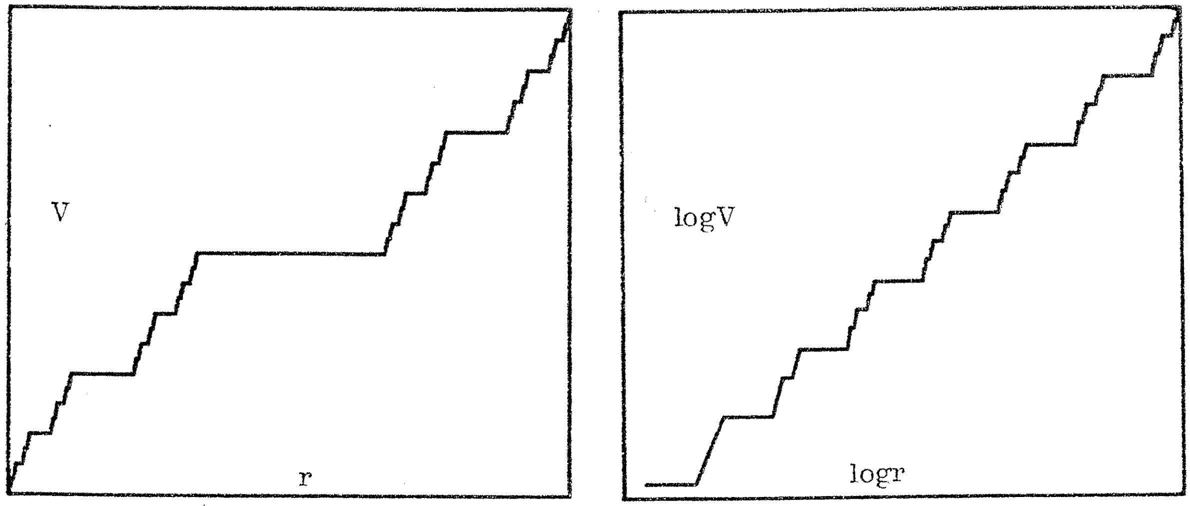

Figure 2

$V_0(r)$ has zero derivative almost everywhere, but $\lim_{r\to 0} \frac{\log V(r)}{\log(r)}$ still exists and is easily calculated to be $\log 2/\log 3$, the Hausdorff dimension of C.

A simple estimate of the type of fluctuations in the dimension estimates which can be expected due to the lack of smoothness in $V_0(r)$ can be obtained from the values $V_0(3^{-n}) = V_0(2\cdot 3^{-n}) = 2^{-n}$. Choosing these two values of r, we obtain the values $\frac{\log 2}{\log 3}$ and $\frac{\log 2}{\log 3 - \frac{1}{n}\log 2} = \frac{\log 2}{\log 3}(1 + \frac{1}{n}\frac{\log 2}{\log 3})$. Thus, fluctuations in the dimension estimates which are associated with the lack of smoothness of $V(r)$ decrease logarithmically with the values of r for which there are good volume estimates. Consequently, an exponentially growing number of points in a random sample are necessary for increased accuracy in dimension estimates constructed from the values of $V(r)$ for two different values of r. I have not explored the possibility that other techniques for estimating $\lim_{r\to 0} \frac{\log V(r)}{\log(r)}$ will yield fluctuations which decrease more rapidly than logarithmic with sample size, but call attention to this issue which may represent a fundamental limitation on the accuracy of numerical estimates of dimension.

Let me turn finally to a discussion of the types of fractal structures which one expects to find in attractors. Since there is no detailed understanding of the fine structure of a "typical" attractor, it is only possible to describe the structure of individual examples and restricted classes of attractors. The issues involved in this discussion are an active area of research, particularly with regard to the conjectures of Yorke et.al. [3] about the relationship between Liapunov exponents and dimensions. Here, I confine attention to two examples, a higher dimensional version of the generalized baker's transformation discussed by Farmer, Ott and Yorke [3] and a quadratic function mapping an interval to itself.

Let I^3 be the unit cube in $\mathbb{R}^3$ and pick numbers $a, b_1, b_2, b_3, b_4 \in (0,1)$. Define a mapping $F\colon I^3 \to I^3$ by requiring that

(1) F is continuous except on the plane $x_1 = a$.

(2) DF is the diagonal matrix with eigenvalues (a^{-1}, b_1, b_2) for $x_1 < a$ and $((1-a^{-1}, b_3, b_4)$ for $x_1 > a$.

(3) F has fixed points at the vertices $(0,0,0)$ and $(1,1,1)$ of I^3.

See Figure 3.

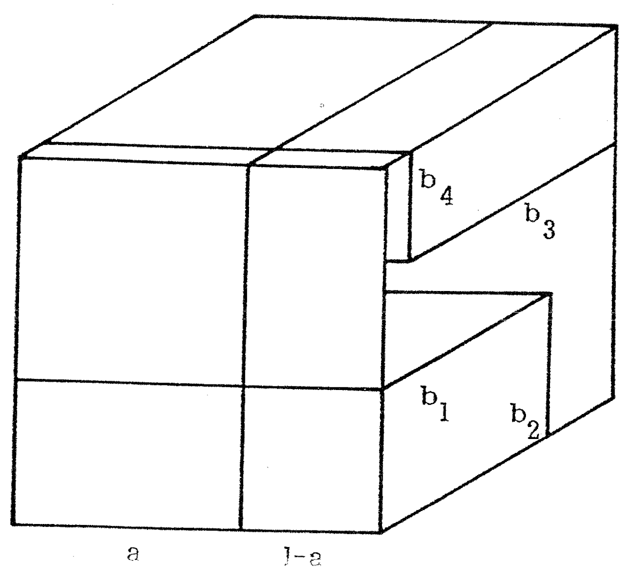

Figure 3

The image of F^n has 2^n components which are rectangular solids, and the mapping DF^n is a diagonal matrix with entries $(a^{-k}(1-a)^{n-k}, b_1^{\,k} b_3^{\,n-k}, b_2^{\,k} b_4^{\,n-k})$ for trajectories which have k iterates satisfying $x_1 < a$. The attractor Λ of F is the product of an interval (in the x_1 direction) with a Cantor set (in the x_2, x_3 directions).

The asymptotic measure for Λ (which describes the asymptotic behavior of almost all trajectories in I^3 with respect to Lebesgue measure) concentrates on a small fraction of the components of $F^n(I^3)$ when n is large and $a \neq \frac{1}{2}$. Almost all trajectories spend a proportion approximately a of their iterates in the region $0 < x_1 < a$ because Lebesgue measure in the x_1 direction is preserved by F.

Thus the asymptotic measure μ is sensitive primarily to components of F^n which have $\frac{k}{n} \approx a$, k being the number of iterates in the region $0 < x_1 < a$. The measure μ is therefore concentrated in rectangular solids whose side lengths are approximately $(b_1^{na} b_3^{n(1-a)}, b_2^{na} b_4^{n(1-a)})$ in the (x_2, x_3) directions. The number of such rectangular solids grows like $(a^{-a}(1-a)^{-(1-a)})^n$. Following Farmer et.al. [3], one can give a complete analysis of the distribution of sizes for the components of $F^n(I^3)$.

Clearly, the dimension of μ will be one plus the dimension of the Cantor set C obtained by intersecting Λ with a plane parallel to the (x_2, x_3) coordinate plane. To study C we have the following construction which yields both C and the measure ν whose product with dx_1 gives μ. Take the square $S = [0,1] \times [0,1]$ and construct two rectangles $R_0 = [0, b_1] \times [0, b_2]$ and $R_1 = [1-b_3, 1] \times [1-b_4]$. The ν measure of R_1 is a and the ν-measure of R_2 is $(1-a)$. Recursively define the Cantor set C and measure ν such that C is contained in 2^n rectangles $R_0^n, \cdots, R^n_{2^n-1}$ and

(1) $R_i^1 = R_i$.

(2) If A_i^n is the affine transformation mapping S onto R_i^n, preserving the coordinate directions with their orientations, then $R_i^{n+1} = A_i^n(R_0)$ and $R^{n+1}_{i+2^n} = A_i^n(R_1)$.

(3) $\nu(R_i^{n+1}) = a\,\nu(R_i^n)$ and $\nu(R^{n+1}_{i+2^n}) = (1-a)\,\nu(R_i^n)$.

See Figure 4.

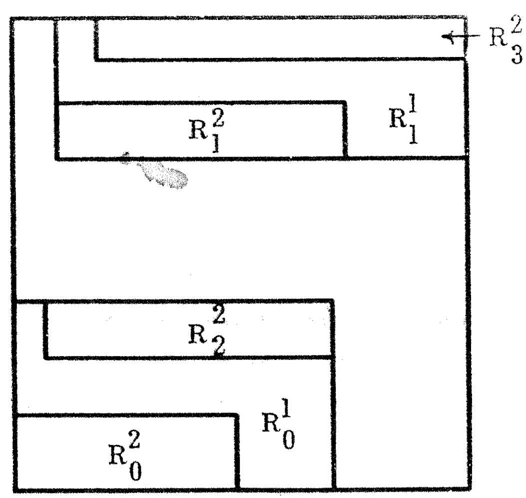

Figure 4

One can do explicit calculations of the volume functions $V_x(r)$ from this description of the Cantor set C. For instance, note that the lower left vertex of R_j^n is located at the point $\left(\sum_{i=1}^{n} c_i b_3^{d_i} b_1^{(i-1-d_i)}(1-b_3)\ ,\ \sum_{i=1}^{n} c_i b_4^{d_i} b_2^{(i-1-d_i)}(1-b_4)\right)$ where $j = \sum_{i=1}^{n} c_i 2^{i-1}$ and $d_i = \sum_{k=1}^{i-1} c_i$. The side lengths of R_j^n are $b_3^{d_n} b_1^{(n-d_n)}$ and $b_4^{d_n} b_2^{(n-d_n)}$ and its ν-measure is $(1-a)^{d_n} a^{(n-d_n)}$. Rather than carrying these calculations farther, I make two remarks: First, the values of $V_x(r)$ will depend strongly on the choice of $x \in C$. Farmer et.al. [3] conjecture that as $r \to 0$, $V_x(r)$ will have a log-normal distribution in x. This dependence of $V_x(r)$ on x must be dealt with sensibly in computing dimension estimates for ν. The second remark is that if $a^a(1-a)^{(1-a)} > b_3^a b_2^{(1-a)} > b_4^a b_2^{(1-a)}$ then the measure will concentrate on rectangles R_j^n whose horizontal and vertical coordinates do not overlap. It follows that the dimension of ν in this case will be independent of b_2 and b_4: $d = a \log \frac{b_3}{a} + (1-a) \log \frac{b_1}{(1-a)}$ since $(a^{-a}(1-a)^{-(1-a)})^n$ disjoint balls of radius $(b_3^a b_1^{(1-a)})^n$ will each have measure approximately $(a^a(1-a)^{(1-a)})^n$. This calculation and the corresponding calculation for the case $b_3^a b_1^{(1-a)} > a^a(1-a)^{(1-a)} > b_4^a b_2^{(1-a)}$ are consistent with the conjectures of Yorke et.al. concerning the relationship between the dimension of ν and the Liapunov exponents of the attractor Λ. Λ.

The second example of an attractor introduced here is a quadratic function $f(x) = a - x^2$ which maps the interval $[-a,a]$ into itself provided that $0 < a \leq 2$. It is known that there is a set $A \subset (0,2]$ of positive Lebesgue measure such that if $a \in A$, then f has an absolutely continuous invariant measure μ whose dimension is 1 [9]. More is known about μ. In particular, μ has a singular density which blows up like $(\pm(x-c))^{\frac{1}{2}}$ to one side of each point c of the form $f^i(0)$, $i > 0$. These singularities will cause large fluctuations in the function $V_x(r)$ as x varies, but the structure of these fluctuations is apparently different from the fluctuations of $V_x(r)$ in the generalized baker's transformation discussed above. The invariant measures of the quadratic transformation are the best available model for the structure that one might expect along expanding directions inside a nonhyperbolic attractor. Since nonhyperbolic attractors are hardly known to exist, only vague speculations about them are possible. Further progress on these matters would be greatly facilitated by additional numerical investigation of examples like the Henon mapping [8].

4. DISCUSSION: FLUID ATTRACTORS. The time has come to consider the analysis of experimental data and the implications of the dimension estimates for a

physical understanding of the dynamics seen in an experiment. A motivating factor is the ability to clearly distinguish physical systems whose dynamics can be described by a low dimensional strange attractor from systems for which such a description is not possible. There are two issues which are intertwined in this problem, namely dimension and determinism. Strange attractors are deterministic: increased precision in one's knowledge of initial state allows one to make predictions about the evaluation of a trajectory for increased time.

It is generally difficult to prepare a physical system so that a specified initial state on a strange attractor can be obtained allowing a direct test for determinism [6]. Instead, one prepares an initial state in the basin of attraction for an attractor but must then wait for transients to die and for the system itself to then follow the attractor until one comes close to the specified state. The sensitivity to initial conditions within the attractor implies that the time at which this will happen is unpredictable and that repetitions of the same experiment will not give reproducible results in this regard. One strategy for obtaining similar initial states is to conduct a long experiment in which a trajectory on an attractor returns close to a state which has already occurred during the experiment. This strategy is reasonable only if the recurrence time is experimentally realistic. If it is too long, then it will be impossible to test whether the systems evolves in a predictable fashion from specified initial conditions for a specified period of time.

The issue of dimension intrudes itself directly into the problem of estimating recurrence times. If one has a reference point x for which one wants to estimate recurrences within a distance r, then the expected recurrence time will be proportional to $(V_x(r))^{-1}$. In a d-dimensional attractor $V_x(r)$ is of order r^d, so that the recurrence time grows exponentially with dimension. For attractors of moderate dimension (say 10), returns within distances of order of 1% should simply not be realizable. Thus an estimate of dimension can be useful in preventing one from attempting the impossible.

Laboratory fluid experiments have been a fertile ground for testing ideas about nonlinear dynamics because one passes from regular to irregular dynamical behavior in regimes where it is difficult to construct a reduced model with a strange attractor from the underlying fluid equations. The experimental results have produced a rich phenomenology, only some of which appears to have a direct analogy with the bifurcations found in low dimensional dynamical systems. Efficient techniques for obtaining crude estimates of dimensions should be a useful probe for studying these issues.

I have tried to separate in this paper those features of the fluctuations in pointwise dimension estimates which are dependent solely upon sampling errors from those which depend upon the complicated geometric structure of attractors. This analysis leads me to assert that the dimension estimates are a practical way of distinguishing (1) situations in which a physical system is quasiperiodic or aperiodic due to weak interactions of several modes that have been excited with comparable amplitude from (2) situations in which there is a low dimensional strange attractor which can be described in terms of few modes. Such a distinction is relevant to experiments such as Rayleigh-Benard convection where the experimental results depend strongly upon the aspect ratio of the container. Experiments with large aspect ratio present a situation in which the fluid instability occurs initially with a large number of modes of the linearized fluid equations being near marginal stability. In this regime, low dimensional attractors have not yet provided good models for the fluid dynamics. Dimension estimates have the potential for demonstrating conclusively that low dimensional models are inappropriate for these fluid regimes.

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