Bifurcations of Relaxation Oscillations

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Abstract

We are still far from a comprehensive theory of bifurcation in dynamical systems with multiple time scales. However, systematic application of geometric methods and study of examples have produced descriptions of varied phenomena. These lectures present a selective summary of some of what has been discovered, concentrating on periodic orbits called relaxation oscillations. We focus upon key aspects of the phenomena, avoiding mathematical details and making the analysis as simple as possible. The dynamics of reduced systems plays a central role in our discussion.

1 Introduction

Relaxation oscillations are periodic orbits that occur in dynamical systems with multiple time scales. They are characterized as periodic orbits in which there are fast and slow segments along the periodic orbit. We formulate these concepts in the context of **slow-fast** vector fields that take the form

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{1.1}$$

or

$$\begin{aligned} x' &= f(x,y) \\ y' &= \varepsilon g(x,y) \end{aligned} \tag{1.2}$$

with $x \in R^m$, $y \in R^n$, $f : R^m \times R^n \to R^m$, and $g : R^m \times R^n \to R^n$. For simplicity, we shall assume throughout these lectures that f and g are C^∞. Here the parameter $\varepsilon \geq 0$ is the ratio of time scales and is assumed to be small. Throughout the paper, we consider families and their solutions that vary continuously as $\varepsilon \to 0$. The limit $\varepsilon = 0$ is singular and looks different for the two systems (1.1) and (1.2). For system (1.1), the limit $\varepsilon = 0$ is a **differential algebraic equation (DAE)** with trajectories evolving on the slow time scale. The system (1.2) emphasizes the fast time scale. The limit $\varepsilon = 0$ is a family of differential

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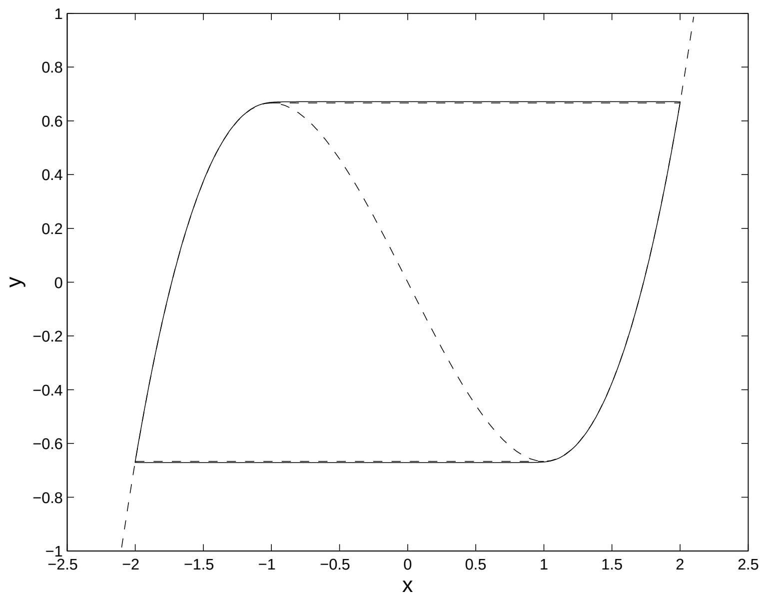

Figure 1: The solid curve is the periodic orbit of the the van der Pol equation (1.3) with $\varepsilon = 0.0001$. The dashed lines are the critical manifold and segments of fast jumps from the folds of the critical manifold. The upper jump segment is difficult to resolve from the periodic orbit at this scale.

equations (called the **layer equations** [29]). The layer equations express the motion of the fast variable x with the variable y acting as a parameter. For fixed y, we say that the equation $x' = f(x, y)$ is a **fast subsystem**.

For (1.1) or (1.2), solutions of the equation $f = 0$ comprise the **critical manifold** of the system. By definition, solutions of the limit DAE of (1.1) lie on the critical manifold. A family of solutions of (1.1) that approach a solution of the DAE as $\varepsilon \to 0$ are called **slow** trajectories and the flow of the DAE is called the **slow flow**. Trajectories of (1.1) that do not lie in a region of phase space where f is $O(1)$ are **fast**. Continuous curves that are a concatenation of slow trajectories and trajectories of the layer equations are called **degenerate phase curves**. The points where a slow segment ends and a fast segment begins is called a **break** point, and the points where a fast segment ends and a slow segment begins is called an **entry** point. **Relaxation oscillations** are periodic orbits whose limits as $\varepsilon \to 0$ are degenerate phase curves with both fast and slow segments.

The term relaxation oscillation was introduced by van der Pol [31] who studied their properties in the vector field that now bears his name:

$$\begin{aligned} \varepsilon\dot{x} &= y - x + x^3/3 \\ \dot{y} &= -x \end{aligned} \tag{1.3}$$

Figure 1 shows the limit cycle of (1.3) with $\varepsilon = 10^{-4}$ together with its critical manifold and a pair of solutions of the layer equation that begin at the fold points of the critical manifold.

We have used the forced van der Pol system

$$\begin{aligned} \varepsilon\dot{x} &= y + x - \frac{x^3}{3} \\ \dot{y} &= -x + a\sin(2\pi\theta) \\ \dot{\theta} &= \omega \end{aligned} \tag{1.4}$$

as a case study to investigate the bifurcations of relaxation oscillations [15, 16, 17]. Although this example does not contain all of the different types of generic bifurcations relaxation oscillations can undergo, it has led to a better understanding of how local degeneracies in the slow-fast decomposition of these periodic orbits play a role in the global bifurcation structure. Since the phenomenon of chaos for dissipative dynamical systems was first observed in the forced van der Pol system, we have paid particular attention to the creation and destruction of chaotic invariant sets. Some of this work is summarized in Section §4 to motivate the discussion of maximal canards that follows in Section §5. Finally, in Section §6 we give a simpler presentation of the "classical" results about canards in another extension of the van der Pol equation.

Our goal is to understand bifurcations of relaxation oscillations as additional parameters (i.e., parameters different from ε) are varied in a slow-fast system. Bifurcations of periodic orbits in generic one-parameter systems with a single time scale have been classified into a small number of types. Specifically, Hopf bifurcations occur as a periodic orbit collapses onto an equilibrium point. Saddle-node, period-doubling (also known as flip) and secondary Hopf (also known as Neimark-Sacker) bifurcations occur where the periodic orbit has neutrally stable eigenvalues and changes stability [18]. Finally, there are homoclinic and saddle-node in cycle bifurcations at which the period of the orbits become infinite. The bifurcations of relaxation oscillations are more varied and harder to analyze numerically than those of periodic orbits in systems with a single time scale. Indeed, our understanding is incomplete, even at the conceptual level of drawing plausible scenarios of how bifurcation occurs for relaxation oscillations. This is especially true for bifurcations associated with some types of qualitative change in the decomposition of orbits into slow and fast segments. These lectures survey what we know and what we conjecture. We emphasize examples and discuss recently discovered phenomena. The material about maximal canards is new, appearing here for the first time. Our presentation does not attempt to be fully rigorous or as general as possible. Instead we try to illuminate the basic principles underlying the bifurcations of relaxation oscillations in their simplest manifestations.

2 Normal Hyperbolicity and Slow Manifolds

We begin our discussion with a review of basic properties of slow-fast dynamical systems (1.1). Formally, we say that $\gamma_\varepsilon : R \times R^+ \to R^m \times R^n$ is a trajectory or solution of (1.1) if

- γ is continuous in ε for $\varepsilon \geq 0$ and smooth in t when $\varepsilon > 0$.
- When $\varepsilon > 0$, $\gamma_\varepsilon(t)$ solves (1.1).
- When $\varepsilon = 0$, $\gamma_0(t)$ is the concatenation of solutions of the DAE (1.1) and the layer equations (1.2).

Arnold et al. [2] say that a curve that satisfies the last statement is a "phase curve of the degenerate system." We emphasize the slow time scale in our treatment of relaxation oscillations, defining below a reduced system that collapses the fast segments to discrete time jumps.

Trajectories of slow-fast systems are often observed to flow along sheets of a critical manifold that are attractors of the layer equations. Jumps along fast segments occur where the trajectory encounters a **singularity** of the projection of the critical manifold onto the space of slow variables y. This prompts a closer examination of the critical manifolds of (1.1) and the singularities of this projection. Sard's theorem [14] states that the regular values of the map $f : R^m \times R^n \to R^m$ are a set of full measure if f is C^n. If 0 is a regular value, then the implicit function theorem implies that the critical manifold C is indeed an n-dimensional manifold. We will assume throughout this paper that this is the case, namely that 0 is a regular value of f and the solutions of $f(x, y) = 0$ constitute a smooth manifold.

The layer equations (1.2) have equilibrium points along the critical manifold. The critical manifold is seldom invariant for the flow of (1.1) when $\varepsilon > 0$, but there are general circumstances in which there is a nearby invariant manifold. The following example illustrates this principle.

Example 2.1:

$$\begin{aligned} \varepsilon\dot{x} &= y - x \\ \dot{y} &= 1 \end{aligned} \tag{2.1}$$

The solutions of this equation are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} y(0) + t - \varepsilon + (x(0) - y(0) + \varepsilon)\exp(-t/\varepsilon) \\ y(0) + t \end{pmatrix}$$

as can be verified by differentiating these expressions with respect to t and observing that $y(t) - x(t) = \varepsilon - (x(0) - y(0) + \varepsilon)\exp(-t/\varepsilon)$. Solving the equation $y(t) = y(0) + t$ for t and substituting $t = y(t) - y(0)$ into the expression for $x(t)$, we find that

$$x(t) = y(t) - \varepsilon + (x(0) - y(0) + \varepsilon)\exp(-(y(t) - y(0)/\varepsilon)$$

on the trajectory with initial point $(x(0), y(0))$. We see that if $x(0) - y(0) + \varepsilon = 0$, then $x(t) = y(t) - \varepsilon$ on the entire solution. Thus the line $y = x + \varepsilon$ is an **invariant manifold** for the vector field (2.1). Moreover, the other solutions of the equation converge to this invariant manifold at the rate $(x(0) - y(0) + \varepsilon)\exp(-t/\varepsilon)$. Note that the invariant manifold is not the solution of the algebraic equation $y - x = 0$ obtained by setting $\varepsilon = 0$ in (2.1), but the distance from the invariant manifold to the line $y = x$ is $O(\varepsilon)$.

The theory of **normally hyperbolicity** [20] has been used by Fenichel [13] to establish the existence of invariant slow manifolds. The fundamental assumption is that the points of the critical manifold are hyperbolic equilibria of their fast subsystems. A technical complication in this theory is that the invariant manifolds have only a finite degree of smoothness, as illustrated by the following example:

Example 2.2:

$$\begin{aligned} \dot{x} &= -r(x + y^r) \\ \dot{y} &= -y \end{aligned} \tag{2.2}$$

The vector field (2.2) is not written explicitly as a slow-fast system, but observe that if the integer r is large, then $1/r$ plays the role of ε in a slow-fast system. The solution of the second equation is $y(t) = y(0)\exp(-t)$. Substituting this into the first equation gives $\dot{x} = -r(x + y^r(0)\exp(-rt))$ which has the solution $x = x(0)\exp(-rt) - ry^r(0)t\exp(-rt)$. Eliminating t from the solutions yields

$$x = x(0)(y/y(0))^r + ry^r \ln(y/y(0))$$

when $y(0) \neq 0$, and $y = 0$ when $y(0) = 0$. The x-axis is a smooth, invariant manifold at the origin along which the motion $x = x(0)\exp(-rt)$ is fast. However, slow invariant manifolds at the origin are constructed by piecing together a pair of curves $x = x(0)(y/y(0))^r + ry^r \ln(y/y(0))$ for two choices of $y(0)$ of opposite sign. The rth derivative of the curves $x = x(0)(y/y(0))^r + ry^r \ln(y/y(0))$ are unbounded as they approach the origin. Thus, the slow manifolds of the system are not infinitely differentiable. Their degree of differentiability increases with r. This illustrates an aspect of normal hyperbolicity: the existence and smoothness of slow manifolds depends upon exponential rates of contraction or expansion normal to the manifold and exponential rates of contraction or expansion along the manifold. Note also that there are many slow manifolds in this system. Indeed, any pair of trajectories on opposite sides of the origin together with the origin give a slow manifold.

These two examples highlight properties of slow manifolds in slow-fast systems that make them elusive. Specifically,

1. Slow manifolds are displaced from the critical manifold by a distance $O(\varepsilon)$.
2. Slow manifolds are not unique
3. Slow manifolds have only a finite degree of smoothness

The existence of invariant slow manifolds was first proved for attracting manifolds in classical work of Tikhonov [26] and for hyperbolic manifolds by Fenichel [13]:

2.1 Theorem *Let $\bar{C}_0$ be a normally hyperbolic, compact manifold with boundary of dimension n contained in the critical manifold of* (1.1)*. There is a continuous family $\bar{C}_\varepsilon$ of C_r invariant manifolds with boundary for* (1.1)*, defined for $\varepsilon \geq 0$ sufficiently small.*

Here invariance of $\bar{C}_\varepsilon$ means that trajectories intersecting $\bar{C}_\varepsilon$ enter or leave $\bar{C}_\varepsilon$ only through its boundary. We remark further that the flow on $\bar{C}_\varepsilon$ is slow and is approximated by the DAE of (1.1).

Associated with the normal hyperbolicity are normalizing "Fenichel" coordinates that make the flow in a neighborhood of the slow manifold preserve a bundle structure. Jones and Kopell introduced a set of normalizing coordinates in their work on the **Exchange Lemma** [22]; a stronger version of these normalizing coordinates was given by Ilyashenko [21]:

2.2 Theorem *Let w be a normally hyperbolic point of the critical manifold of* (1.1) *at which the slow vector field defined by g is non-zero. For any $r > 0$ and $\varepsilon > 0$ sufficiently small, there is a neighborhood of w and a C^r change of coordinates that transforms* (1.1) *into a system of the form*

$$\begin{aligned} \varepsilon\dot{x} &= a(y,\varepsilon)x \\ \dot{y} &= 1 \end{aligned} \tag{2.3}$$

The DAE of (1.1) can be reduced to a system of ordinary differential equations on normally hyperbolic critical manifolds. The Fenichel Theorem implies that this system of ODE's is a good approximation for the slow flow along invariant manifolds of (1.1). Thus we take the ODE's on the critical manifold as the backbone for defining a **reduced system** of (1.1) that describes its limit properties as $\varepsilon \to 0$. However, trajectories may depart from the critical manifold at points where the projection of the slow flow onto the space of slow variables y is singular. How this happens is discussed in the next section.

3 Singularities, Folds and Jumps

Denote by $\pi : R^m \times R^n \to R^n$ the projection $\pi(x, y) = y$. Normal hyperbolicity of the critical manifold C fails at singular points of $\pi|C$, the restriction of the projection to C. We shall denote the set of these singular points by C_s. The equation $\det(D_x f) = 0$ is a **defining equation** for C_s. Observe that the DAE of (1.1) may fail to have solutions at points of C_s as illustrated by the following example.

Example 3.1:

$$\begin{aligned} \varepsilon \dot{x} &= y - x^2 \\ \dot{y} &= -1 \end{aligned} \tag{3.1}$$

When $\varepsilon = 0$, the origin is a minimum of y along the curve $y - x^2 = 0$, but the differential equation says that $\dot{y} = -1$. These two conditions are clearly inconsistent since $\dot{y}$ must be zero to remain on the curve $y - x^2 = 0$. Thus the DAE does not always give an immediate representation of limiting solutions to (1.1) as $\varepsilon \to 0$. We can ask under which circumstances there is a unique way to continue solutions of the DAE at singularities of its critical manifold so as to give the limit behavior of solutions of the slow-fast system (1.1). To deal systematically with this issue, we turn to singularity theory and bifurcation theory.

Singularity theory [14] gives a set of tools for classifying the singularities of smooth maps up to smooth coordinate changes. Arnold et al. [2] discuss application of singularity theory to the critical manifolds of generic slow-fast systems having two slow variables. Generic maps $h : R^2 \to R^2$ have only two types of singularities: **folds** and **cusps**. A similar result holds for the critical manifolds of generic slow-fast systems with two slow variables. Folds are characterized by the condition that $D_x f$ has corank one and an inequality on second derivatives of f: if w and v are the left and right eigenvectors of $D_x f$, then $w D_{xx} f(v, v) \neq 0$. At folds, there is a coordinate change of the form $(u, v) = (h(x, y), k(y))$ that maps the critical manifold to the set of solutions of $u_1 - v_1^2 = 0$; $u_i = 0, 2 \leq i \leq m$, but these coordinate changes do not preserve the flow of (1.1). Even in dynamical systems with a single time scale, questions concerning equivalence of vector fields by smooth change of coordinates at equilibrium points are intricate, involving resonances [18] and small divisors [1]. In the case of slow-fast systems, every point of the critical manifold is an equilibrium point of its fast subsystem, and these questions become still more complicated. There has been little progress thus far in classifying slow-fast systems at points of C_s up to smooth equivalence.

Bifurcation theory for single time scale systems can be applied to the fast dynamics of (1.2). At fold points which are saddle-node equilibria of their fast subsystem with no eigenvalues having a positive real part, there is a single unstable separatrix leaving the saddle-node point [18]. If this separatrix approaches another equilibrium of the fast subsystem as its ω-limit, then the fast time motion of the trajectory starting at the fold point tends to the unstable separatrix as $\varepsilon \to 0$. In the limit, we define a discrete (slow) time **jump** for the reduced system that maps the fold point to the ω-limit of its unstable separatrix.

More formally, we define the reduced system of a slow-fast system (1.1) in the context in which the limit sets of the fast subsystems are all equilibria. If this assumption holds, we say that the slow-fast system has **no rapid oscillations**. We define the **reduced system** of a slow-fast system with no rapid oscillations on its critical manifold by allowing a point w to evolve in one of two ways: it may follow the slow flow of the DAE, or if there is a fast trajectory with α-limit w and ω-limit z, then it may jump in discrete (slow) time from w to z. This definition allows the system to be multi-valued, so the uniqueness property for the evolution of a dynamical system breaks down. For example, at regular points of an unstable sheet of the critical manifold, there is the option of evolving along the slow flow or jumping along a trajectory of the unstable manifold in the fast subsystem. A comprehensive treatment of reduced systems requires reconsideration of basic properties of dynamical systems, something we do not pursue here. Nonetheless, this definition of the reduced system is a very useful tool for the study of bifurcations of relaxation oscillations.

There is an important situation in which the reduced system is single valued, namely when slow trajectories flow along stable sheets of the critical manifold, reach its boundary transversally at saddle-node points of the fast subsystem and then jump to a new stable sheet of the critical manifold. Nearby trajectories also have these properties. The smoothness of the jump maps has been studied by Levinson [23], Pontryagin [27] and Szmolyan and Wechselberger [30]. The fundamental result is that, in the circumstances described above, phase curves of the degenerate system are the limits of trajectories of the slow-fast system (1.1) as $\varepsilon \to 0$. If these hypotheses are met, then there are asymptotic expansions that describe the ε dependence of how the slow-fast trajectories approach the degenerate phase curve. As will be evident below, the trajectories do not depend smoothly on ε, complicating the analysis of the jump phenomenon. We incorporate these concepts into the following definition, adding additional requirements on the entry points of trajectories at a jump.

A relaxation oscillation is **simple** if the slow-fast decomposition of its degenerate phase curve γ_0 satisfies the following properties:

- The slow segments of γ_0 lie in the closure of the normally hyperbolic, stable sheets of the critical manifold C.
- The terminal points of the slow segments lie on the boundary of the normally hyperbolic, stable sheets of the critical manifold C and approach the boundary C_s of C transversally.
- The terminal points of the slow segments are saddle-node points of their fast subsystems.
- The initial points of the slow segments are in the interior of the normally hyperbolic, stable sheets of the critical manifold C.
- If the image J of a jump map defined on C_s contains an initial point w, then the slow vector field at w is transverse to J in C_s.

Simple relaxation oscillations that persist remain simple when the slow-fast systems (1.1) is perturbed. They may encounter bifurcations similar to those present in systems with a single time scale, A key part of the strategy that we advocate to explore bifurcations of relaxation oscillations is to examine how generic one parameter families of relaxation oscillations cross the boundary of the region of simple relaxation oscillations. This can occur through several different mechanisms, so a goal becomes to investigate these.

In generic slow-fast systems (1.1) with two slow variables ($n = 2$) and one fast variable ($m = 1$), C_s is a curve. The flow of a slow-fast system near generic break points can be transformed by an ε dependent change of coordinates to a **system of first approximation** [2] of the form

$$\begin{aligned} \dot{x} &= y + x^2 + O(\mu) \\ \dot{y} &= 1 + O(\mu) \\ \dot{z} &= x + O(\mu) \end{aligned} \tag{3.2}$$

with

$$x = O(\mu) \qquad y = O(\mu^2) \qquad z = O(\mu^3) \qquad t = O(\mu^2) \qquad \varepsilon = \mu^3$$

The limit as $\varepsilon \to 0$ of (3.2) clearly converges. The limit system is invariant under translation along the z-axis and the motion in the (x, y) plane is independent of z. In the (x, y) plane there is a unique trajectory that remains a finite distance from the half parabola $y+x^2$, $x < 0$. This trajectory divides a region of trajectories whose x coordinates tend to $-\infty$ in finite decreasing time from a region which tends to the half parabola $y + x^2$, $x > 0$ with decreasing time. The dividing trajectories are the rescaled limit of the trajectories that flow along the stable slow manifold. Their x-coordinates tend to ∞ in finite time, with a limit value of $y > 0$. Since y is scaled by $\varepsilon^{2/3}$ in the system of first approximation, the asymptotic expansions of the flow past a generic fold are not smooth in ε. We shall discuss the system (3.2) further in Section §3 when we consider the properties of maximal canards.

At fold points, coordinates for the critical manifold must mix the slow and fast coordinates since the projection π onto the space of slow coordinates is singular. Since the gradient of f is non-zero, $\partial f/\partial y_1$ or $\partial f/\partial y_2$ is non-zero, say $\partial f/\partial y_1 \neq 0$. This implies that there is a function $h(x, y_2)$ so that the critical manifold near the fold point is locally given by $y_1 = h(x, y_2)$. The slow flow can be expressed in terms of (x, y_2) coordinates so that, after rescaling, it extends to the fold curve. The equation $f(x, y) = 0$ is differentiated along trajectories to give the relationship

$$\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y_1}\dot{y_1} + \frac{\partial f}{\partial y_2}\dot{y_2} = 0$$

on the critical manifold. Since $\partial f/\partial y_1 \neq 0$, this equation can be solved for $\dot{y_1}$:

$$\dot{y_1} = -(\frac{\partial f}{\partial y_1})^{-1}(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y_2}\dot{y_2})$$

Substituting this into $\dot{y} = g$ and gives

$$-(\frac{\partial f}{\partial y_1})^{-1}\frac{\partial f}{\partial x}\dot{x} = g_1 + (\frac{\partial f}{\partial y_1})^{-1}\frac{\partial f}{\partial y_2}g_2$$

Rescaling time, we obtain the system

$$\begin{aligned} x' &= g_1 + (\frac{\partial f}{\partial y_1})^{-1}\frac{\partial f}{\partial y_2}g_2 \\ y_2' &= -(\frac{\partial f}{\partial y_1})^{-1}\frac{\partial f}{\partial x}g_2 \end{aligned} \tag{3.3}$$

where the functions (f, g_1, g_2) are evaluated at $(x, h(x, y_2), y_2)$. We continue to call (3.3) the slow flow. This system has equilibrium points on the fold curve when $g_1 = 0$ since $\partial f/\partial x = 0$ defines the fold curve. Generically, g_1 vanishes at isolated points of the fold curve. These points are called **folded equilibria**. They are not limits of equilibrium points of the full system. Instead they occur at locations where the flow is tangent to the fold curve, trajectories flowing toward the fold curve on one side of the folded equilibrium and away from the fold curve on the other side of the folded equilibrium.

The dynamics of slow-fast systems (1.1) near generic folded equilibria was first studied by Benoit [4]. He proved the existence of **canards**, trajectories that continue along the unstable sheet of the critical manifold after approaching the folded equilibrium, in the cases that the equilibrium is a saddle or node. The geometry of the flow near a folded node has not been fully characterized and remains as an open problem in the theory of slow-fast systems. The analysis of folded equilibria proceeds in several steps. First, preliminary coordinate changes and rescalings are performed to produce a system of first approximation that is closely related to the **normal form**

$$\begin{aligned} \varepsilon\dot{x} &= y - x^2 \\ \dot{y} &= az + bx \\ \dot{z} &= 1 \end{aligned} \tag{3.4}$$

With the scalings

$$x = O(\mu) \qquad y = O(\mu^2) \qquad z = O(\mu) \qquad t = O(\mu) \qquad \varepsilon = \mu^2$$

used to obtain the system of first approximation, the normal form retains its form exactly with $\varepsilon = 1$. There is a folded equilibrium at the origin.

The slow flow of the normal form (3.4) after its time rescaling is

$$\begin{aligned} x' &= az + bx \\ z' &= 2x \end{aligned} \tag{3.5}$$

The system (3.5) is linear. If $a > 0$, then the origin is a saddle; if $a < 0$, then the origin is a node if $b^2 + 8a > 0$ and a focus if $b^2 + 8a < 0$. A node or focus is stable if $b < 0$ and unstable if $b > 0$. In the case of a saddle or a node, there are a pair of invariant lines for the slow flow along the eigenvectors of the system. These lines are limits of a pair of polynomial solutions of the full system (3.4) as $\varepsilon \to 0$. The polynomial solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}(t) = \begin{pmatrix} \alpha t \\ \varepsilon\alpha + \alpha^2 t^2 \\ t \end{pmatrix} \tag{3.6}$$

where $\alpha = (b \pm \sqrt{b^2 + 8a})/4$ is a root of the equation $2\alpha^2 - b\alpha - a = 0$.

We now analyze the case of a folded saddle further. When the origin is a folded saddle, the two values of alpha have opposite sign. The polynomial solution of the full system with $\alpha = (b - \sqrt{b^2 + 8a})/4 < 0$ is a **maximal** canard that remains on the slow manifold for all time, flowing from its stable sheet to its unstable sheet. The polynomial solution with $\alpha = (b + \sqrt{b^2 + 8a})/4 > 0$ also lies on the slow manifold, but flows from its unstable sheet to its stable sheet. It has been called a "faux-canard" by Benoit [4].

Note that the distance between the two polynomial solutions is $2\varepsilon\alpha = O(\varepsilon)$. Solutions attracted to the stable sheet of the slow manifold approach it at a rate $O(\exp(-c/\varepsilon))$, except in the vicinity of the fold curve. Therefore, we expect that trajectories with canards will be close enough to the maximal canard that a variational analysis along this canard will describe their general properties. We are especially interested in computing a transition map for trajectories that approach the maximal canard on the stable sheet of the slow manifold and then jump away from the unstable sheet of the slow manifold after following a canard segment for some distance.

The equations (3.4) have a time reversal symmetry $R(x, y, z, t) = (-x, y, -z, -t)$, so the process of jumping from the unstable sheet of the slow manifold is a mirror image of the process by which trajectories approach the stable sheet of the slow manifold. The stable and unstable slow manifolds are not quite two dimensional surfaces, but rather three dimensional sets of thickness $O(\exp(-c/\varepsilon))$ in directions parallel to the fast flow. We want to demonstrate that stable and unstable slow manifolds containing the maximal canard intersect transversally as they reach the plane $z = 0$. Our argument is based upon an analysis of the variational equations of the flow along the maximal canard.

The variational equations along a trajectory $w(t) : R \to R^3$ of (3.4) are given by $\dot{\xi}(t) = A(w(t))\xi(t)$ where A is the matrix

$$A = \begin{pmatrix} -2x/\varepsilon & 1\varepsilon & 0 \\ b & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

Along the maximal canard trajectory

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}(t) = \begin{pmatrix} \alpha t \\ \varepsilon\alpha + \alpha^2 t^2 \\ t \end{pmatrix} \tag{3.8}$$

$x = \alpha t$ in the equation (3.7) with $\alpha = (b - \sqrt{b^2 + 8a})/4 < 0$. The equation becomes simpler if we make a change of coordinates along the trajectory. Set

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2\alpha t \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \tag{3.9}$$

and $\xi = B\eta$. The equation (3.7) transforms to $\dot{\eta} = B^{-1}(AB - \dot{B})\eta$. Now

$$B^{-1} = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -2\alpha^2 t \\ 0 & 0 & \alpha \end{pmatrix} \tag{3.10}$$

giving

$$\dot{\eta} = \begin{pmatrix} \frac{-2\alpha t}{\varepsilon} & \frac{1}{\varepsilon} & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \eta \tag{3.11}$$

Thus the vector $\eta(t) = (0, 0, 1)^t$ is one solution of the equation (3.11), and the subspace spanned by the first two unit vectors is invariant. We want to follow the evolution of vectors in this subspace, paying particular attention to their direction at $t = 0$. Assuming that $b < 0$, the rescaling

$$\zeta_1 = \sqrt{-b\varepsilon}\eta_1 \qquad \zeta_2 = \eta_2 \qquad \tau = \sqrt{-b/\varepsilon}$$

produces the system

$$\zeta' = \begin{pmatrix} 2\gamma\tau & 1 \\ -1 & 0 \end{pmatrix} \zeta \tag{3.12}$$

which has the single parameter $\gamma = \alpha/b$. The time derivative in (3.12) is with respect to τ. Note that in the case of the folded saddle, $\gamma = \alpha/b > 1/2$ for the negative root $\alpha = (b - \sqrt{b^2 + 8a})/4$ and $b < 0$. If we set $\theta = \arg(\zeta)$, then

$$\begin{aligned} \theta' = \frac{\zeta_2'\zeta_1 - \zeta_1'\zeta_2}{\zeta_1^2 + \zeta_2^2} &= -\cos^2(\theta) - 2\gamma\tau\cos(\theta)\sin(\theta) - \sin^2(\theta) \\ &= -1 - 2\gamma\tau\cos(\theta)\sin(\theta) = -1 - \gamma\tau\sin(2\theta) \end{aligned} \tag{3.13}$$

We want to follow θ as τ approaches 0. For most initial conditions, with $\tau \ll 0$, $\theta(0)$ will give the direction of the intersection of the stable slow manifold with the plane $z = 0$ in the ζ coordinates. Using the time reversal symmetry R, the comparable direction for the intersection of the unstable slow manifold with $z = 0$ will be $-\theta(0)$. Therefore, if $\theta(0)$ is not a multiple of $\pi/2$, the stable and unstable slow manifolds will intersect transversally at $z = 0$.

Let us turn to the analysis of the equation $\theta' = -1 - \gamma\tau\sin(2\theta)$. The equation is π-periodic in θ, so we focus attention on $\theta \in [0, \pi]$. The zeros of the right hand side occur when $\tau = -1/(\gamma\sin(2\theta))$. In the interval $(-1/\gamma, 0]$, $\theta' < 0$. When $\tau < -1/\gamma$, $\theta' = 0$ on the curve $\tau = -1/(\gamma\sin(2\theta))$ and $\theta' < 0$ when $\theta = \pi/2$. Consequently, trajectories that enter the θ interval between the upper branch of $\tau = -1/(\gamma\sin(2\theta))$ and $\theta = \pi/2$, remain there until they reach $\tau = -1/\gamma$. Most trajectories with initial conditions $\tau \ll 0$ (fixed) and $\pi/4 < \theta < \pi$ (varying) quickly converge to one another with θ close to $\pi/2$. These trajectories arrive at $\tau = -1/\gamma$ with $\pi/4 < \theta < \pi/2$. Now $\partial\theta'/\partial\gamma = -\tau\sin(2\theta) > 0$ when $\tau < 0$ and $0 < \theta < \pi$. Therefore, if the trajectory with initial condition $\tau \ll 0$ and $\theta = \pi/2$ reaches $\tau = 0$ with $\theta > 0$, then this will be true for all trajectories with the same initial conditions and larger values of γ.

We use numerical integration to estimate for which value of γ, trajectories with initial conditions $\tau \ll 0$ and $0 < \theta < \pi$ pass through the origin. These calculations suggest that the critical value of γ is 1/2. In the case of the folded saddle, $\gamma > 1/2$ so we find that $\theta(0) > 0$ for these trajectories, implying transversality of the slow stable and unstable manifolds.

The presence of canards near folded saddles complicates the definition of the reduced system for (1.1). Canard trajectories can jump from the unstable sheet of the critical manifold anywhere along the saddle separatrix of the slow flow. Thus the reduced system becomes multivalued: the trajectory arriving along the stable manifold of the saddle in the slow flow may continue through the saddle. At any point along this stable manifold on the unstable sheet of the critical manifold, the trajectory may do one of three things: continue on the unstable sheet or jump to one of the two sides of the critical manifold along the fast direction until it hits a new sheet of the critical manifold. To understand the effects associated with this behavior better, we turn to the forced van der Pol equation.

4 The Forced van der Pol Equation

The phase space of the forced van der Pol system (1.4) is $R^2 \times S^1$ and its critical manifold is the surface defined by $y = x^3/3 - x$. The singularities of the critical manifold are the curves defined by $x = \pm 1$, $y = \mp 2/3$. The slow flow is given by

$$\begin{aligned} \theta' &= \omega(x^2 - 1) \\ x' &= -x + a\sin(2\pi\theta). \end{aligned} \tag{4.1}$$

The reduced system is given by this slow flow (4.1) together with jumps from the fold curves $x = \pm 1$ to $x = \mp 2$. Folded singularities of the reduced system occur when $x = a\sin(2\pi\theta) = \pm 1$. Thus, folded singularities occur in the parameter regime $a \geq 1$. There is a narrow strip of the (a, ω) parameter space bounded by $a = 1$ where there are folded nodes, but the folded singularities are saddles and foci in most of this parameter plane. At the folded saddles, we augment the reduced system with trajectories representing the canards and their jumps. The folded foci do not play a substantial role in the dynamics of the system because the unstable manifolds of the saddles of (4.1) "shield" the foci from trajectories that have undergone a jump: trajectories of the slow flow with initial conditions on $x = \pm 2$ do not reach the folded foci in the parameter regimes we consider.

Our approach to studying the dynamics of the forced van der Pol system has been based upon analysis of return maps for its reduced system [17]. We utilize the symmetry $s(x, y, \theta) = (-x, -y, \theta + 0.5)$ in defining the return map. Slow flow trajectories with initial conditions on $x = 2$ flow until they hit the fold at $x = 1$. Note that if $a > 2$, x first increases on some of these trajectories and there are points where the final requirement for a relaxation oscillation to be simple is violated. Trajectories jump to $x = -2$ when they reach $x = 1$ unless they are on the stable manifold of the folded saddle or node and continue along canards for some distance before they jump. To incorporate the canards most directly into the return map, we define cross-sections to the jumps as suggested by Weckesser [6]. The cross sections are the half-cylinders Σ_1 and Σ_2 defined by $x = 1,\ y > -2/3$ and $x = -1,\ y < 2/3$. A **half-return map** σ is obtained by flowing from Σ_1 to Σ_2 and then applying the symmetry s to get back from Σ_2 to Σ_1. Except for the canards, the image of σ lies along the circle $y = -2/3$. The canards adjoin to the image of σ two additional segments: one from jumps of canards from the unstable sheet of the critical manifold directly to Σ_2 as x decreases and one from jumps of canards back to the stable sheet of the critical manifold with $x > 0$, where they return to the fold and jump a second time to Σ_2. The image of the second segment also lies on the circle $y = -2/3$, while the image of the first segment does not. The map σ has rank 1: when points jump from Σ_1 to the critical manifold with $x > 1$, those that land on the same trajectory of the slow flow have the same σ image. Thus we idealize σ as a one dimensional map with the images of the points in the stable manifold of the folded saddle being entire intervals. Chaotic invariant sets formed from trajectories with canards are present in the full system (1.4) when the images of the canard segments of the reduced system contain a point of the stable manifold of the folded saddle. The analysis of folded saddles in Section §3 gives information about the scale of these chaotic invariant sets. This analysis breaks down when the canard trajectories that return to the stable manifold are **maximal**, meaning that the break point of the canard is at the place that the saddle separatrix returns to the fold curve on the critical manifold. Figure 2 depicts the stable manifold of the folded saddle and the

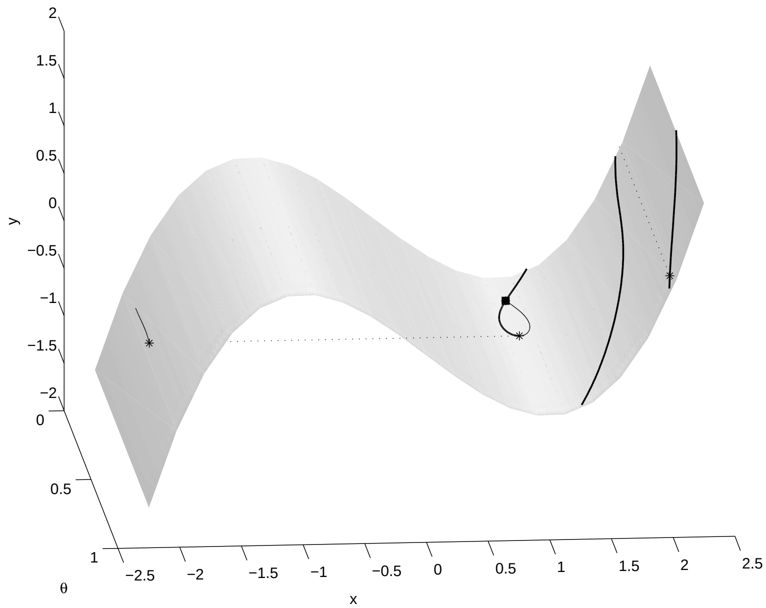

Figure 2: The critical manifold of the forced van der Pol equation (1.4) is drawn as a shaded surface together with the stable manifold and canards of the folded saddle as thick solid curves. The θ direction is periodic with fundamental domain $[0, 1]$ shown. The images of the canards after jumping to the stable sheets of the critical manifold are drawn as thin solid curves. The jump from the maximal canard and the line $(x, y) = (2, 2/3)$ are drawn as dotted curves. The folded saddle is marked by a square; the maximal canard, its image after jumping to the critical manifold at $x = -2$, and the image of this point under the symmetry s are marked by asterisks. The parameters are $(a, \omega) = (1.1, 1.48)$.

maximal canard of the slow flow at parameter values for which the maximal canard flows into the stable manifold.

5 Maximal Canards

The creation of horseshoes in families of two dimensional maps has been extensively studied, with the Henon map serving as a prototype [19]. The principal results [3, 32] extend Jakobson's Theorem for one dimensional maps [8] into the setting of two dimensional diffeomorphisms with small Jacobian. We have examined canards in vector fields with two slow and one fast variable. Here we focus upon "maximal" canards that reach a fold, connecting two families of canards that jump in opposite directions from an unstable slow manifold. In the vicinity of the maximal canards we study, the vector field has a return map with a fold and small Jacobian. The singular limit of the return map has some aspects that differ from the Henon family. In particular, the critical point of the limit map is not C^3: it has different order from the two sides and different limit directions. The scenario that we describe below for the creation of chaotic invariant sets takes place in the forced van der Pol system.

Consider a one parameter family of slow-fast vector fields with two slow variables and one fast variable that has the following properties:

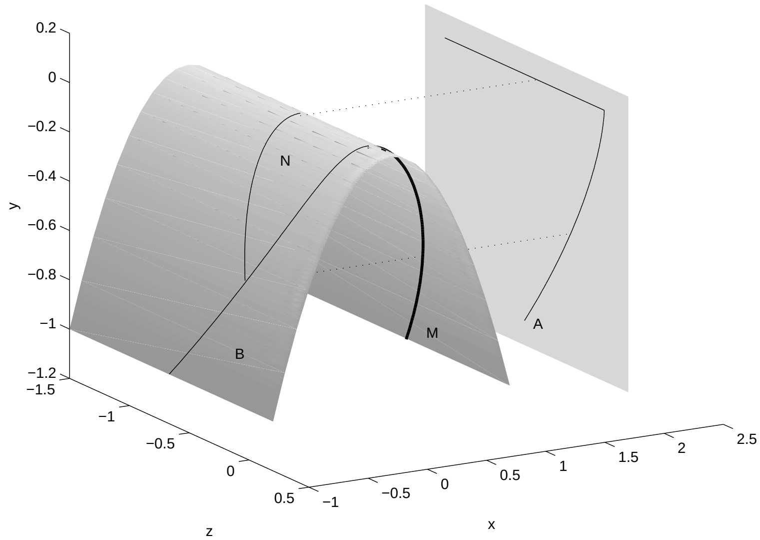

Figure 3: The reduced flow of the system (3.2) showing the behavior of maximal canards at a fold. The canard trajectory is the heavy trajectory labeled M. The critical manifold and a cross-section to the jumps are drawn shaded. The images of the canard after jumps are solid curves A and B. Dotted lines are shown along jumps from one point along the canard. From the points that jump back to the critical manifold, the reduced trajectories flow along the stable sheet of the critical manifold to its fold (solid curve N) where it jumps to the cross-section (dotted line). The image on the cross-section consists of a segment parallel to the fold of the critical manifold and the image of the jumps that proceed directly from the canard to the cross-section.

- There is a two dimensional critical manifold with a smooth fold curve S and a folded saddle p_s.
- In the reduced system, the canard separatrix of p_s on the unstable sheet of the critical manifold intersects the fold curve S transversally at a point p_m, the *maximal canard point*. The point p_m is a regular fold point.
- There is a value of the parameter so that the trajectory of p_m in the reduced system flows to p_s, all jumps occurring at regular fold points. As the parameter varies, the trajectory of p_m in the reduced system crosses the stable manifold of p_s transversally.
- Strong contraction dominates strong expansion along maximal canard trajectories beginning and ending near p_s.

Figure 3 depicts the maximal canard trajectories.

Our first objective is to describe in the reduced system the limiting behavior of trajectories near the maximal canards. The maximal canard solution reaches a fold and then jumps, but it does so by following the unstable sheet of the slow manifold. The system of first approximation (3.2) for generic trajectories reaching a fold gives a model for the maximal canard. We derive the reduced system for (3.2) by setting $\varepsilon = 0$, differentiating $y + x^2 = 0$

with respect to time, substituting $-2x\dot{x}$ for $\dot{y}$ and then rescaling time to obtain

$$\begin{aligned} \dot{x} &= 1 \\ \dot{z} &= -2x^2 \end{aligned} \tag{5.1}$$

The equation (5.1) is easily integrated: we normalize the trajectories $(x(t), z(t)) = (t, c - 2t^3/3)$ so that they reach the fold curve at zero time t. Thus the trajectory beginning at (x_0, z_0) has $t = x_0$ and $c = z_0 + 2x_0^3/3$. If $x_0 > 0$, we flow trajectories with decreasing time since the time rescaling used to obtain the reduced system is negative on the unstable sheet of the critical manifold. On the critical manifold, $y(t) = -x^2(t)$ so each solution projects onto a semicubical parabola $(-t^2, c - 2t^3/3))$ in the (x, z) plane.

All of the solutions of (3.2) are unbounded, tending to infinity in finite time along the direction parallel to the x-axis. We want to see how trajectories in the reduced system follow the canard for varying distances, jump and then intersect a cross-section $x = b > 0$. Without loss of generality, assume that the canard trajectory has $c = 0$ on the unstable sheet $x > 0$ of the critical manifold. Using the formula we computed above, this trajectory is given by the curve $(t, -t^2, -2t^3/3)$ on the critical manifold with $t > 0$. Note that t decreases along the solution curve. If the trajectory jumps in the direction of increasing x at time $t = u$, then it hits the cross-section $x = b > 0$ at the point with coordinates $(-u^2, -2u^3/3)$. If the trajectory jumps in the direction of decreasing x at time $t = u$, then it lands on the stable sheet of the critical manifold at the point $(-u, -2u^3/3)$. The reduced trajectory through the point $(-u, -2u^3/3)$ is $(t, -4u^3/3 - 2t^3/3)$ with $t = -u < 0$ initially. The trajectory then flows along the stable sheet of the critical manifold until it reaches the fold at $x = y = 0$ at time $t = 0$ and with $z = -4u^3/3$. From the fold, it jumps parallel to the x-axis and hits the cross-section $x = b$ at the point $(0, -4u^3/3)$. Thus the intersection of the trajectories jumping from canards on the cross-section $x = b$ traces out a curve C which is not smooth at the point coming from the maximal canard. It has one segment along the z-axis and one segment lies along a semicubical parabola tangent to the y-axis.

We return to the study of the flow map σ from a small cross-section $y = a$ of the canard trajectories to the plane $x = b$ for the full system (3.2). We expect the image to lie close to the curve C described above. Both the reduced and full system are translation invariant in the z direction, so we examine the flow of an initial segment S parallel to the x axis that intersects the canard trajectory. Since $\dot{y} = 1$, the evolution of the x coordinate is given by the equation $\varepsilon\dot{x} = x^2 + a + t$. The motion of the initial points that concern us is to flow along the slow manifold $x^2 + y = 0$ to a height $y = u$ where the trajectory jumps from the slow manifold in the direction of increasing or decreasing x. In the direction of increasing x, the trajectory proceeds nearly parallel to the x axis until it hits the plane $x = b$ near $(-u^2, -2u^3/3)$. In the direction of decreasing x, the trajectory approaches the stable sheet of the slow manifold, flows to the fold and then flows nearly parallel to the x axis, hitting the plane $x = b$ near $(0, -4u^3/3)$. We need to estimate u, the y-coordinate of the break point, in terms of the distance of the initial point from the slow manifold. Let x_s be the x coordinate of a point on the intersection of the canard orbit with the section $y = a$. If $x - x_s = v$ is small, then the deviation of the trajectories on the section $y = a$ from the slow manifold can be estimated by solving the variational equation $\varepsilon\dot{\xi} = 2x(t)\xi$ with initial condition v and $x(t) = \sqrt{-(a + t)}$ the x coordinate of the canard trajectory of the reduced system. The solution of this equation is

$$\xi(t) = v\exp\left(\frac{2\int_0^t \sqrt{-(a+\tau)}d\tau}{\varepsilon}\right) = v\exp\left(\frac{4}{3\varepsilon}((-a)^{3/2} - (-(a+t))^{3/2}\right)$$

The jump height can be approximated by solving $|\xi(t)| = 1$ and then using $u = a + t$. This gives an estimate u_e for the value of u at the break point as

$$u_e = -\left(\frac{3\varepsilon}{4}\log|v| + (-a)^{3/2}\right)^{2/3}$$

Note that the derivative

$$\frac{du_e}{dv} = \frac{-\varepsilon}{2v}\left(\frac{3\varepsilon}{4}\log|v| + (-a)^{3/2}\right)^{-1/3}$$

is unbounded as $|v| \to 0$. This calculation implies that σ stretches S.

In the vicinity of the maximal canard, we require finer analysis of σ. We apply the inverse of the scale transformation $(x, y, z, t) = (\varepsilon^{1/3}X, \varepsilon^{2/3}Y, \varepsilon Z, \varepsilon^{1/3}T)$ to determine the scale of the domain near the maximal canard where more analysis of the system of first approximation (3.2) is needed. We want to see that $\sigma(S)$ is a strictly convex curve near the maximal canard, so that the methods used for the study of Henon-like maps [32] can be applied in this setting. Explicit or asymptotic solutions to (3.2) near the maximal canard are not apparent, so we rely upon numerical integration to study its solutions. Figure 4 shows a plot of $\sigma(S)$ with $a = -4$, $b = 10$ and the x coordinates of S lying in the interval $[1.93111, 1.93143]$. The circles are the images of 32 points regularly spaced on S at increments of 10^{-5}. The crosses are the values of the least squares fit to this data by a sixth degree polynomial $y(x) \approx 6.91 + 0.693x - 0.265x^2 - 0.113x^3 - 0.0172x^4 + 0.080x^5 - 0.0395x^6$. The second derivative of this approximating polynomial is negative: the maximum of its second derivative is approximately -0.398. These data are strong evidence for the convexity of $\sigma(S)$.

Based on this analysis of the flow near canards of (3.2), we give a conjectural description of the generic properties of flow past a maximal canard that ends in a generic manner near a fold. Let R_0 be a rectangle transverse to a canard trajectory, with one side parallel to the unstable sheet of the slow manifold and one side parallel to the strong expanding direction. Let R_1 be a section transverse to the fast trajectory which begins at the end p of a maximal canard. We want to examine the flow σ from R_0 to R_1. In the direction parallel to the slow manifold on R_0, the dynamics remain slow and there is no fast expansion or contraction prior to a jump. Segments S parallel to the strong expanding direction on R_0 are mapped to R_1 by σ in the following manner. Exponentially short pieces of S follow the unstable slow manifold long enough to contain canards. These short segments are exponentially stretched by an amount that can be estimated from integration of the variational equation for the fast expansion. Denoting by $p_{s,\varepsilon}$ points of S whose trajectories reach the fold of the slow manifold, the limiting image of $\sigma(S)$ as $\varepsilon \to 0$ has a corner at $\sigma(p_{s,0})$ and different asymptotics for jumps in opposite directions. In terms of distance to $\sigma(p_{s,0})$, $\sigma(S)$ is quadratic for jumps directly to R_1 and cubic for jumps back to the stable sheet of the slow manifold and then to R_1. This behavior breaks down in a neighborhood of $\sigma(p_{s,\varepsilon})$ of size comparable to $\varepsilon^{2/3}$. In this region $\sigma(S)$ is a smooth, strictly convex curve. With a scaling that magnifies this region by amounts $(\varepsilon^{-2/3}, \varepsilon^{-1})$ transverse and parallel to the image of the fold, $\sigma(S)$ has a smooth limit as $\varepsilon \to 0$.

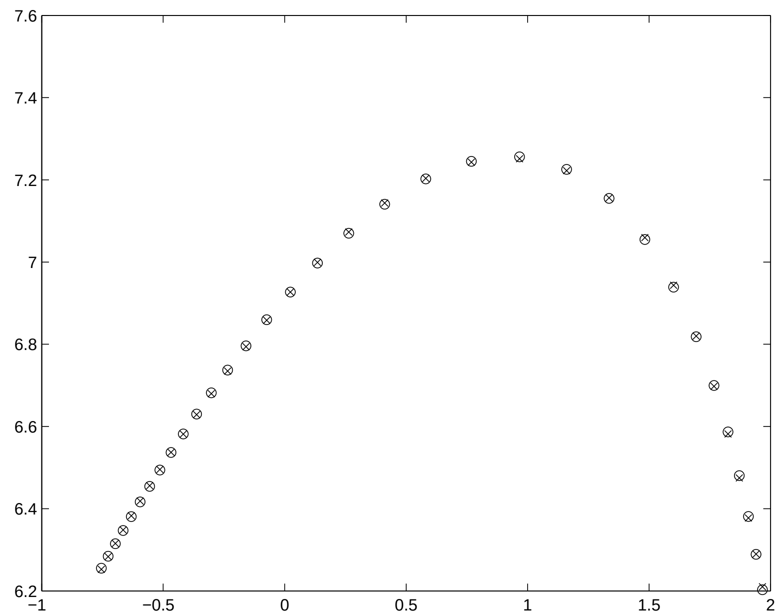

Figure 4: The graph of $\sigma(S)$ on the cross-section. Data points obtained from numerical integration are marked by an $\circ$ while the least squares fit by a polynomial of degree six are marked by $\times$.

6 "Classical" canards

A group in Strasbourg [5, 9] discovered canards in the two dimensional vector fields

$$\begin{aligned} \varepsilon\dot{x} &= y - (\frac{x^3}{3} - x) \\ \dot{y} &= a - x \end{aligned} \tag{6.1}$$

There are Hopf bifurcations that occur at $a = \pm 1$ in this extension of the van der Pol equation. The analysis of this system was originally done with techniques of non-standard analysis and then repeated using asymptotic methods [12] and geometric singular perturbation theory [11]. Here we shall apply simpler geometric methods to recover basic properties of this example.

We begin with a coordinate transformation that fixes the equilibrium point of the system at the origin. Set

$$\begin{aligned} u &= x - a \\ v &= y - (\frac{a^3}{3} - a) \\ a &= b + 1 \end{aligned} \tag{6.2}$$

to obtain

$$\begin{aligned} \varepsilon\dot{u} &= v - (\frac{u^3}{3} + (b+1)u^2 + (b^2 + 2b)u) \\ \dot{v} &= -u \end{aligned} \tag{6.3}$$

The system (6.3) has a Hopf bifurcation at $b = 0$. The formula for the relevant cubic coefficient of its normal form shows that the bifurcation is supercritical, with stable limit cycles appearing for $b < 0$. To study how these limit cycles grow, we consider the *rotational* properties of the vector fields as b varies.

6.1 Theorem *(Duff [10]) Let X_b be a one parameter family of planar vector fields such that*

- *X_b has a stable, hyperbolic limit cycle γ_0 that is oriented clockwise when $b = b_0$*
- *$X \times \frac{\partial X}{\partial b}$ is negative on γ.*

Then there is a neighborhood B of b_0, such that $b \in B$ implies that X_b has a stable, hyperbolic limit cycle γ_b. The curves γ_b are disjoint and shrink with increasing b.

Taking the right-hand side of (6.3) by X, we have

$$X \times \frac{\partial X}{\partial b} = \frac{-1}{\varepsilon}(u^3 + (2b+2)u^2)$$

This is negative in the half plane $u > -(2b+2)$. Applying Duff's theorem, we conclude that the limit cycles grow monotonically with decreasing b, at least until they come close to the line $u = -2$.

Further insight into the flow of (6.3) can be obtained by studying its slow manifolds. We are especially interested in two aspects of these manifolds: (1) the amount of expansion and contraction that occurs and (2) where the stable and unstable parts of the manifold intersect the v coordinate axis. To address (1), we use the variational equation for fast contraction and expansion along the critical manifold. The fast contraction and expansion of (6.3) takes place along the u direction, with a neighborhood of the origin to be avoided. The variational equation in this direction is $\varepsilon\dot{\xi} = -(u^2 + 2(b+1)u + b^2 + 2b)\xi$. We want to compute solutions to this equation along relaxation oscillations approximated by stable and unstable segments of the critical manifold, together with jumps parallel to the u-axis. The solution of the variational equation is

$$\xi(t) = \xi(0)\exp\left(\frac{-1}{\varepsilon}\int_0^t (u^2 + 2(b+1)u + b^2 + 2b)dt\right)$$

We change the variable of integration to u via $dt = \frac{dt}{dv}\frac{dv}{du}du$ and use the estimate that $\frac{dv}{du} \approx u^2 + 2(b+1)u + b^2 + 2b$ along the regular portions of the critical manifold. When $b = 0$, the integral becomes $\int -u(u+2)^2 du = -(u^4/4 + 4u^3/3 + 2u^2)$. We compute the integral on the portion of the critical manifold lying over u intervals (u_0, u_1) oriented with decreasing u where $-2 < u_1 < 0 < u_0 < 1$ and $u_0^3/3 - u_0 = u_1^3/3 - u_1$. These intervals correspond to the slow segments of a "duck without head," a relaxation oscillation containing a canard in which the jump from the canard is with increasing u. We find that the integrals are all positive. Figure 5 plots the integrals as a function of u_0. We conclude from this computation that the ducks without heads are all strongly attracting. The ducks with heads (relaxation oscillations with decreasing u along jumps from the canards) are even more strongly attracting.

Fenichel theory breaks down in the vicinity of the origin because normal hyperbolicity of the critical manifold fails there. Nonetheless, trajectories with initial conditions near the stable part of the critical manifold with $u > 0$ cross $u = 0$ in a tiny, exponentially small

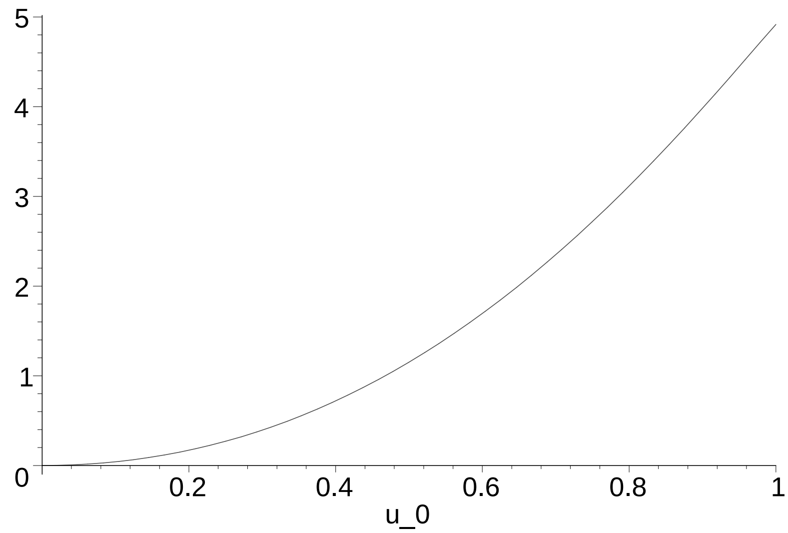

Figure 5: Values of the variational integral that give the magnitude of the exponential contraction for canards without heads as a function of their right hand end point u_0 in system (6.3).

interval I_s. The same is true of backward trajectories with initial conditions near the unstable part of the critical manifold with $-1 < u < 0$, giving an interval I_u. Relaxation oscillations with canards will only occur when I_u and I_s intersect. Thus, we want to estimate the location of these intervals and determine how they vary with b. This is done by locating pieces of the slow manifold of the system within a strip of width $O(\varepsilon^2)$.

We observe first that the curve

$$v = h_0(u) = \frac{u^3}{3} + (b+1)u^2 + (b^2+2b)u - \varepsilon\frac{u}{u^2+2(b+1)u+b^2+2b}$$

has the property that

$$\frac{dh_0}{du} - \frac{\dot v}{\dot u} = \varepsilon\frac{u^2-b^2-2b}{(u+b)^2(u+b+2)^2}$$

For $(1+b) < u$, this difference is positive, implying that h is transverse to the vector field and, since $\dot u < 0$, the trajectories of the vector field cross the graph of h_0 from right to left. However, on the curve $v = h_1(u) = h_0(u) + \varepsilon^2$ with $b = 0$, we have

$$\frac{dh_1}{du} - \frac{\dot v}{\dot u} = -\varepsilon(\frac{u^5+8u^4+24u^3+32u^2+16u-1}{(u+2)^2}) + O(\varepsilon^2)$$

This quantity is negative when $0.056 < u < 1$, so trajectories cross the graph of h_1 from left to right in this interval of values of u. When b and ε are sufficiently small, trajectories entering the strip between the graphs of h_0 and h_1 remain there until u decreases to 0.06. Now h_0 has a non-zero derivative with respect to b, so the slow manifold varies at a non-zero rate with respect to b. In particular, its intersection with $u = 0$ has a negative derivative with respect to b.

A similar calculation to the one above demonstrates that the unstable segment of the slow manifold for u in a compact subinterval of $(-2, 0)$ lies in the strip between the graph of

h_0 and $h_2 = h_0(u) - \varepsilon^2$. When $b = 0$ we compute

$$\frac{dh_2}{du} - \frac{\dot{v}}{\dot{u}} = \varepsilon\left(\frac{u^5 + 8u^4 + 24u^3 + 32u^2 + 16u - 1}{(u+2)^2}\right) + O(\varepsilon^2)$$

Next we observe that when $\varepsilon = 0$, the sign of $\frac{dh_0}{db} = u(u+2) + O(b)$ is positive for $u > 0$ and negative for $u < 0$. From this we conclude that the intersection of the right branch of the slow manifold with $u = 0$ decreases at a non-zero rate as b decreases, while the intersection of the unstable, middle branch of the slow manifold with $u = 0$ increases at a non-zero rate as b decreases. Consequently, there is only an exponentially small range of b in which a stable slow manifold from the right can meet an unstable slow manifold. At the initial points of the family of canard orbits, as v decreases on a vertical line the jump points first move from the origin to the maximal canard point near $u = -2$ with jumps toward increasing u; then the jump points return toward the origin with jumps toward decreasing u. The jump points on the periodic orbits with canards vary in the same way as b decreases. This establishes the basic properties of the family of periodic orbits with canards in system (6.1).

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