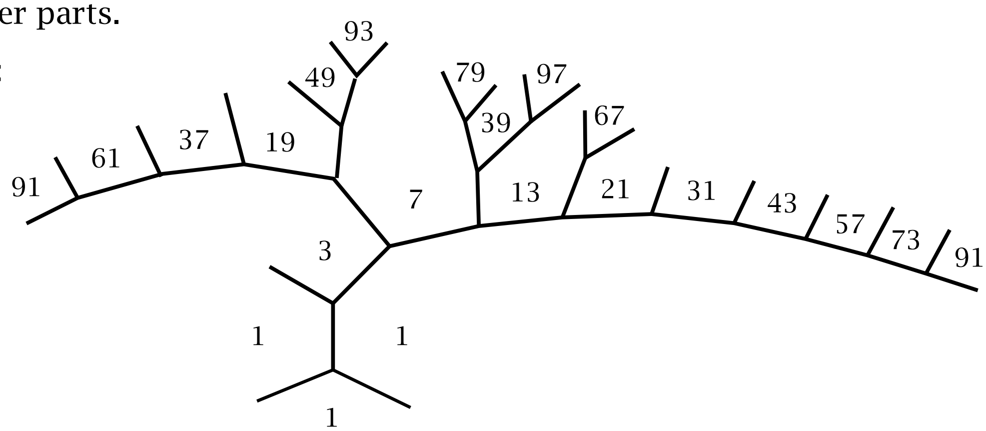
1. For the form $Q(x,y) = x^2 + xy + y^2$ do the following things:

(a) Draw enough of the topograph to show all the values less than 100 that occur in the topograph. You do not need to draw parts of the topograph that are symmetric with other parts.

Solution:



(b) Make a list of the primes less than 100 that occur in the topograph, and a list of the primes less than 100 that do not occur.

Solution: The primes that occur are $3, 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97$.

The primes that do not occur are $2, 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89$.

(c) Characterize the primes in the two lists in part (b) in terms of congruence classes modulo $|\Delta|$ where Δ is the discriminant of Q.

Solution: We have $\Delta = -3$. The primes that occur are 3 and the primes congruent to 1 mod 3. The primes that do not occur are the primes congruent to -1 mod 3.

(d) Characterize the nonprime values in the topograph in terms of their factorizations into primes in the lists in part (b).

Solution: The nonprime values that occur are the products of the prime values that occur, except that the factor 3 can only occur to the first power, so 3^2, 3^3, 3^4, and $3^2 \cdot 7$ do not occur (but 7^2 does occur).

(e) Summarize the previous parts by giving a simple criterion for which numbers are representable by the form Q, i.e., the numbers n such that $Q(x,y) = n$ has an integer solution (x,y), primitive or not. The criterion should say something like n is representable if and only if $n = m^2 p_1 \cdots p_k$ where each p_i is a prime such that ...

Solution: The numbers represented are the numbers $n = m^2 p_1 \cdots p_k$ where each p_i is either 3 or a prime congruent to 1 modulo 3.

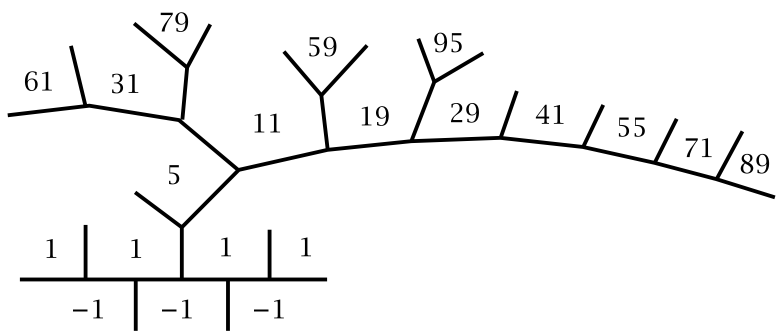
(f) Check that all forms having the same discriminant as Q are equivalent to Q.

Solution: This was part of an earlier problem set. We have $4ac = h^2 + D = h^2 + 3$ with $h^2 \le D/3 = 1$ so there is only one possibility:

h	ac	(a,c)
1	1	$(1,1)$

2. Do the same things for the form $x^2 + xy - y^2$. This form is hyperbolic and it takes the same negative values as positive values, so you can just ignore all the negative values.

Solution:

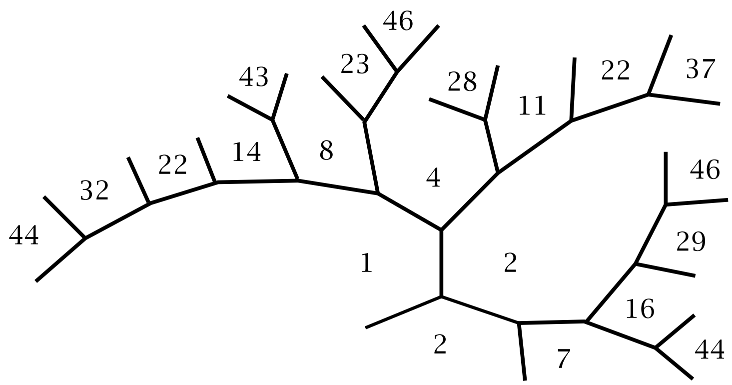


Here $\Delta = 5$. The primes in the topograph are $5, 11, 19, 29, 31, 41, 59, 61, 71, 79, 89$. Apart from 5, these are the primes congruent to ± 1 modulo 5. The primes that do not occur are $2, 3, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97$. These are the primes congruent to 2 or 3 modulo 5. The nonprimes that occur are the products of the primes that occur, except that the factor 5 can occur only to the first power. The numbers represented by this form are the numbers $n = m^2 p_1 \cdots p_k$ where each p_i is either 5 or a prime congruent to ± 1 modulo 5. There is only one equivalence class of forms for $\Delta = 5$, as we see from the table of solutions to $\Delta = 5 = h^2 + 4pq$:

h	pq	(p,q)
1	1	$(1,1)$

3. Do the same things for the form $x^2 + xy + 2y^2$, except that this time you only need to consider values less than 50 instead of 100.

Solution:



Here $\Delta = -7$. The primes in the topograph are $2, 7, 11, 23, 29, 37, 43$. Apart from 7, these are the primes congruent to $1, 2, 4$ modulo 7. The primes that do not occur are $3, 5, 13, 17, 19, 31, 41, 47$. These are the primes congruent to $3, 5, 6$ modulo 7. The nonprimes that occur are the products of the primes that occur, except that the factor 7 can occur only to the first power. The numbers represented by this form are the numbers $n = m^2 p_1 \cdots p_k$ where each p_i is either 7 or a prime congruent to $1, 2, 4$ modulo 7. There is only one equivalence class of forms for $\Delta = -7$, as we see from the table of solutions to $4ac = h^2 + 7$, $h^2 \le 7/3$:

h	ac	(a,c)
1	2	$(1,2)$

4. For discriminant $\Delta = -24$ do the following:

(a) Verify that the class number is 2 and find two quadratic forms Q_1 and Q_2 of discriminant -24 that are not equivalent.

Solution: We have $4ac = h^2 + 24$ and $h^2 \le D/3 = 8$.

h	ac	(a,c)
0	6	$(1,6)$, $(2,3)$

The corresponding forms are $Q_1 = x^2 + 6y^2$ and $Q_2 = 2x^2 + 3y^2$.

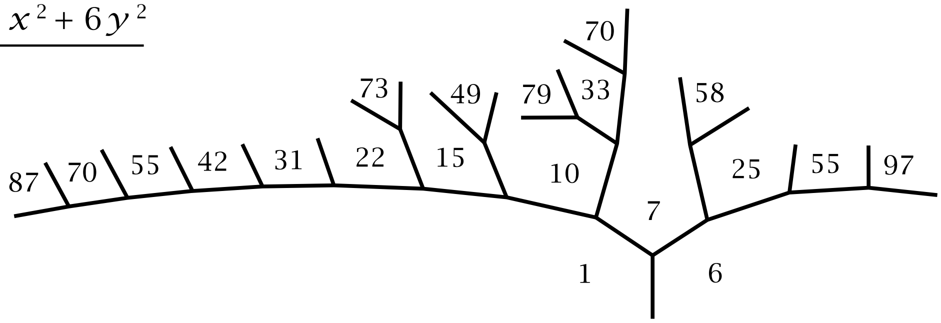
(b) Draw topographs for Q_1 and Q_2 showing all values less than 100.

Solution: These are shown on the next page.

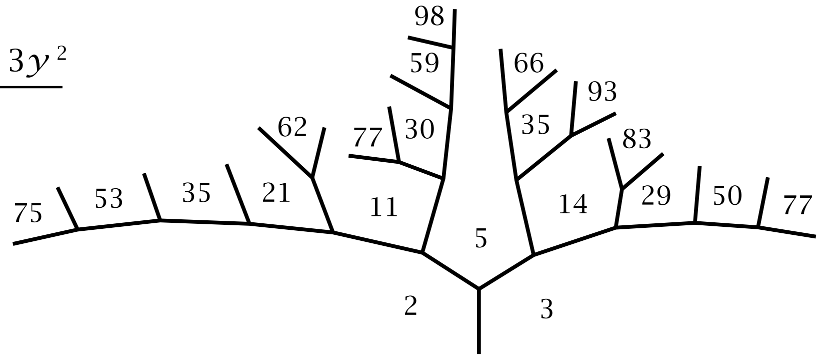
(c) Divide the primes less than 100 into three lists: those represented by Q_1, those represented by Q_2, and those represented by neither Q_1 nor Q_2.

Solution: The primes represented by Q_1 are $7, 31, 73, 79, 97$. The primes represented by Q_2 are $2, 3, 5, 11, 29, 53, 59, 83$. The primes represented by neither form are $13, 17, 19, 23, 37, 41, 43, 47, 61, 67, 71, 89$.

$Q_1(x, y) = x^2 + 6y^2$



$Q_2(x, y) = 2x^2 + 3y^2$



(d) Characterize the primes in the three lists in part (c) in terms of congruence classes modulo $|\Delta| = 24$.

Solution: The primes represented by Q_1 are the primes congruent to 1 or 7 mod 24. The primes represented by Q_2 are 2, 3, and the primes congruent to 5 or 11 mod 24. The primes not represented by Q_1 or Q_2 are the primes congruent to $13, 17, 19, 23$ mod 24.

(e) Characterize the nonprime values in the topograph of Q_1 in terms of their factorizations into primes in the lists in part (c), and then do the same thing for Q_2. Your answers should be in terms of whether there are an even or an odd number of prime factors from certain of the lists.

Solution: For Q_1 the nonprimes in the topograph are products of primes represented by Q_1 and an even number of primes represented by Q_2. For Q_2 the nonprimes in the topograph are products of primes represented by Q_1 and an odd number of primes represented by Q_2.

(f) Summarize the previous parts by giving a criterion for which numbers are representable by the form Q_1 and which are representable by Q_2.

Solution: The numbers represented by Q_1 are the numbers $n = m^2 p_1 \cdots p_k$ where each p_i is a prime represented by either Q_1 (i.e. congruent to 1 or 7 mod 24) or by Q_2 (i.e either 2 or 3 or congruent to 5 or 11 mod 24), with an even number of p_i's represented by Q_2. For Q_2 the answer is the same except that the number of p_i's represented by Q_2 is odd.

5. This problem will show how things can be more complicated than in the previous problems.

(a) Show that the class number for discriminant -23 is 2 and find forms Q_1 and Q_2 of discriminant -23 that are not equivalent.

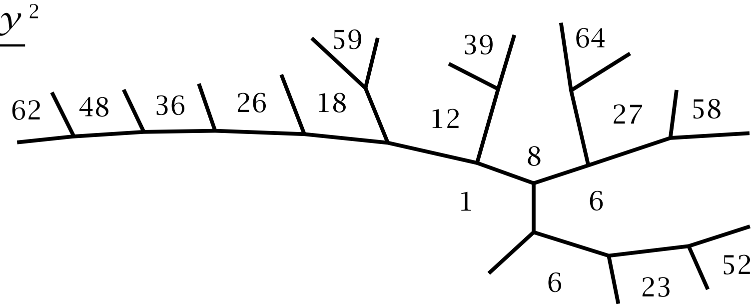
Solution: We have $4ac = h^2 + 23$ and $h^2 \le 23/3$.

h	ac	(a,c)
1	6	$(1,6)$, $(2,3)$

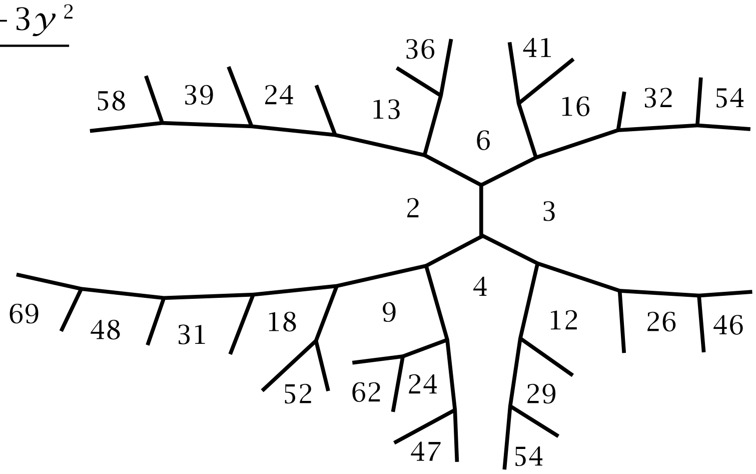
The corresponding forms are $Q_1 = x^2 + xy + 6y^2$ and $Q_2 = 2x^2 + xy + 3y^2$.

(b) Draw the topographs of Q_1 and Q_2 up to the value 70.

$Q_1(x,y) = x^2 + xy + 6y^2$



$Q_2(x,y) = 2x^2 + xy + 3y^2$



(c) Find a number n that occurs in both topographs, and find the x and y values that give $Q_1(x_1, y_1) = n = Q_2(x_2, y_2)$. (This sort of thing never happens in the previous problems.)

Solution: 6 is in both topographs as $Q_1(0,1)$ and $Q_2(1,1)$. Also 39 is in both topographs as $Q_1(3,2)$ and $Q_2(4,1)$.

(d) Find a prime p_1 in the topograph of Q_1 and a different prime p_2 in the topograph of Q_2 such that p_1 and p_2 are congruent modulo $|\Delta| = 23$. (This sort of thing also never happens in the previous problems.)

Solution: The primes 59 in the topograph of Q_1 and 13 in the topograph of Q_2 are congruent mod 23.