1. (a) Show that if α and β are elements of $\mathbb{Z}[\sqrt{D}]$ such that α is a unit times β, then $N(\alpha) = \pm N(\beta)$.

(b) Either prove or give a counterexample to the following statement: If α and β are Gaussian integers with $N(\alpha) = N(\beta)$ then α is a unit times β.

2. Show that a Gaussian integer $x + yi$ with both x and y odd is divisible by $1 + i$ but not by $(1 + i)^2$.

3. There are four different ways to write the number $1105 = 5 \cdot 13 \cdot 17$ as a sum of two squares. Find these four ways using the factorization of 1105 into primes in $\mathbb{Z}[i]$. [Here we are not counting $5^2 + 2^2$ and $2^2 + 5^2$ as different ways of expressing 29 as the sum of two squares. Note that an equation $n = a^2 + b^2$ is equivalent to an equation $n = (a + bi)(a - bi)$.]

4. (a) Find four different units in $\mathbb{Z}[\sqrt{3}]$ that are positive real numbers, and find four that are negative.

(b) Do the same for $\mathbb{Z}[\sqrt{11}]$.

5. Make a list of all the Gaussian primes $x + yi$ with $-7 \le x \le 7$ and $-7 \le y \le 7$. (The only actual work here is to figure out the primes $x + yi$ with $0 \le y \le x \le 7$, then the rest are obtainable from these by symmetry properties.)

6. Factor the following Gaussian integers into primes in $\mathbb{Z}[i]$: $3 + 5i$, $8 - i$, $10 + i$, $5 - 12i$, $35i$, $-35 + 120i$, $253 + 204i$.

7. In this problem we consider $\mathbb{Z}\sqrt{-2}$. To simplify notation, let $\omega = \sqrt{-2}$, so elements of $\mathbb{Z}[\omega]$ are sums $x + y\omega$ with $x, y \in \mathbb{Z}$ and with $\omega^2 = -2$. We have $N(x + y\omega) = x^2 + 2y^2 = (x + y\omega)(x - y\omega)$.

(a) Draw the topograph of $x^2 + 2y^2$ including all values less than 70 (by symmetry, it suffices to draw just the upper half of the topograph). Circle the values that are prime (prime in $\mathbb{Z}$, that is). Also label each region with its x/y fraction.

(b) Which primes in $\mathbb{Z}$ factor in $\mathbb{Z}[\omega]$?

(c) Using the information in part (a), list all primes in $\mathbb{Z}[\omega]$ of norm less than 70.

(d) Draw a diagram in the xy-plane showing all elements $x + y\omega$ in $\mathbb{Z}[\omega]$ of norm less than 70 as small dots, with larger dots or squares for the elements that are prime in $\mathbb{Z}[\omega]$. (There is symmetry, so the primes in the first quadrant determine the primes in the other quadrants.)

(e) Show that the only primes $x + y\omega$ in $\mathbb{Z}[\omega]$ with x even are $\pm\omega$. (Your diagram in part (d) should give some evidence that this is true.)

(f) Factor $4 + \omega$ into primes in $\mathbb{Z}[\omega]$.