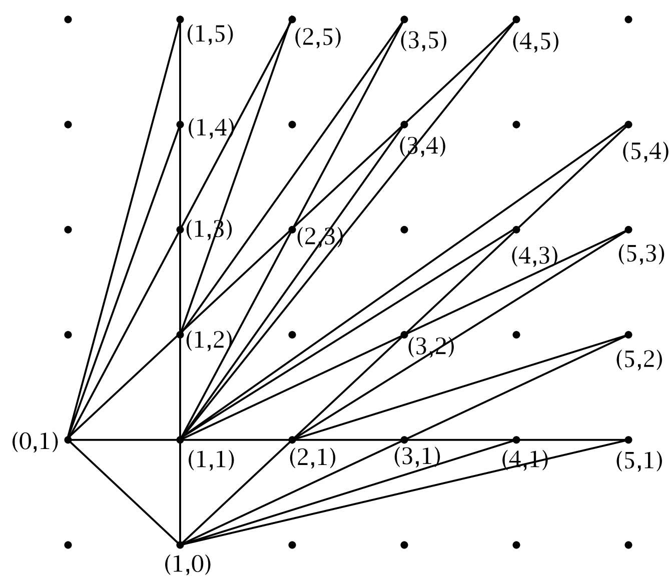
1. This problem involves another version of the Farey diagram, or at least the positive part of the diagram, the part consisting of the triangles whose vertices are labeled by fractions p/q with $p \geq 0$ and $q \geq 0$. In this variant of the diagram the vertex labeled p/q is placed at the point (q, p) in the plane. (Thus p/q is the slope of the line through the origin and (q, p).) The edges of the new Farey diagram are straight line segments connecting the pairs of vertices that are connected in the original Farey diagram. Thus for example there is a triangle with vertices $(1, 0)$, $(0, 1)$, and $(1, 1)$ corresponding to the big triangle in the upper half of the circular Farey diagram.
What you are asked to do in this problem is just to draw the portion of the new Farey diagram consisting of all the triangles whose vertices (q, p) satisfy $0 \leq q \leq 5$ and $0 \leq p \leq 5$. Note that since fractions p/q labeling vertices are always in lowest terms, the points (q, p) such that q and p have a common divisor greater than 1 are not vertices of the diagram.
Solution:



2. Compute the Farey series F_{10}.
Solution: We can start with F_7 which is in the book:

$$\frac{0}{1}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{1}{1}$$

To get F_8 we add $1/8, 3/8, 5/8, 7/8$ to get

$$\frac{0}{1}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{5}{8}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{1}{1}$$

Next we add $1/9, 2/9, 4/9, 5/9, 7/9, 8/9$ to get

$$\frac{0}{1}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{1}{2}\ \frac{5}{9}\ \frac{4}{7}\ \frac{3}{5}\ \frac{5}{8}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{7}{9}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{8}{9}\ \frac{1}{1}$$

Finally we add $1/10, 3/10, 7/10, 9/10$:

$$\frac{0}{1}\ \frac{1}{10}\ \frac{1}{9}\ \frac{1}{8}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{2}{9}\ \frac{1}{4}\ \frac{2}{7}\ \frac{3}{10}\ \frac{1}{3}\ \frac{3}{8}\ \frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}\ \frac{1}{2}\ \frac{5}{9}\ \frac{4}{7}\ \frac{3}{5}\ \frac{5}{8}\ \frac{2}{3}\ \frac{7}{10}\ \frac{5}{7}\ \frac{3}{4}\ \frac{7}{9}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{7}{8}\ \frac{8}{9}\ \frac{9}{10}\ \frac{1}{1}$$

3. (a) Compute the values of the continued fractions ${}^1\!\nearrow_1 + {}^1\!\nearrow_3 + {}^1\!\nearrow_5 + {}^1\!\nearrow_7$ and ${}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_2$.

__Solution__: For ${}^1\!\nearrow_1 + {}^1\!\nearrow_3 + {}^1\!\nearrow_5 + {}^1\!\nearrow_7$ we start with $5 + 1/7 = 36/7$. Next we invert and add 3: $3 + 7/36 = 115/36$. Now we invert and add 1: $1 + 36/115 = 151/115$. Finally we invert to get $115/151$.

For ${}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_1 + {}^1\!\nearrow_2$ we have $1 + 1/2 = 3/2$, $1 + 2/3 = 5/3$, $1 + 3/5 = 8/5$, $1 + 5/8 = 13/8$, $1 + 8/13 = 21/13$ and finally inverting this gives $13/21$. Notice that the numbers occurring here are Fibonacci numbers.

(b) Compute the continued fraction expansions of $19/44$ and $101/1020$.

__Solution__: For $19/44$ we compute $44/19 = 2 + 6/19$ and then $19/6 = 3 + 1/6$ so the answer is $19/44 = {}^1\!\nearrow_2 + {}^1\!\nearrow_3 + {}^1\!\nearrow_6$.

For $101/1020$ we have $1020/101 = 10 + 10/101$ and $101/10 = 10 + 1/10$ so the answer is $101/1020 = {}^1\!\nearrow_{10} + {}^1\!\nearrow_{10} + {}^1\!\nearrow_{10}$.

4. (a) Compute the continued fraction for $38/83$ and display the steps of the Euclidean algorithm as a sequence of equations involving just integers.

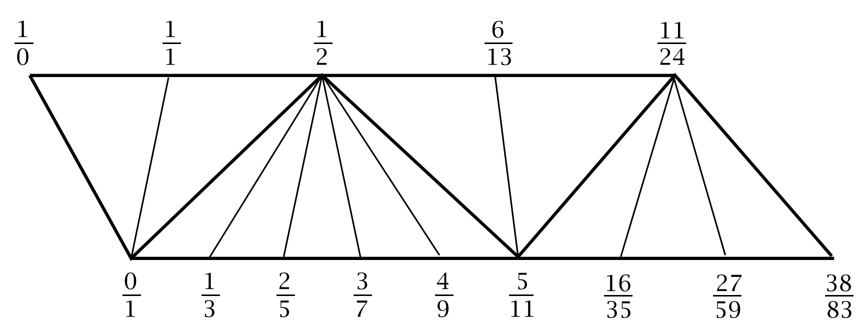
__Solution__: We have

$$\begin{aligned} 83 &= \underline{2} \cdot 38 + 7 \\ 38 &= \underline{5} \cdot 7 + 3 \\ 7 &= \underline{2} \cdot 3 + 1 \\ 3 &= \underline{3} \cdot 1 \end{aligned}$$

The underlined numbers give the continued fraction $38/83 = {}^1\!\nearrow_2 + {}^1\!\nearrow_5 + {}^1\!\nearrow_2 + {}^1\!\nearrow_3$.

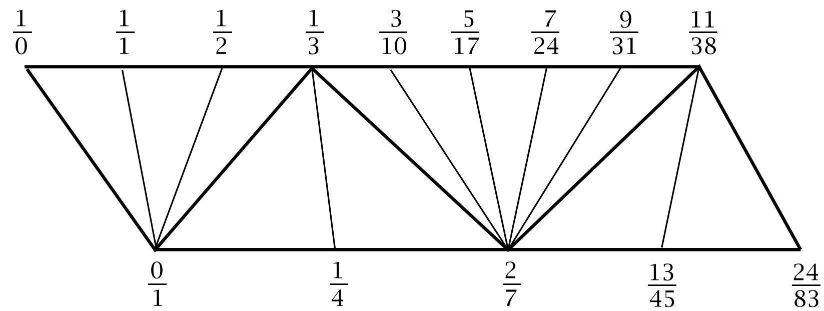
(b) For the same number $38/83$ compute the associated strip of triangles (with large triangles subdivided into fans of smaller triangles), including the labeling of the vertices of all the triangles.

__Solution__:



(c) Take the continued fraction $1/a_1 + 1/a_2 + \cdots + 1/a_n$ you got in part (a) and reverse the order of the numbers a_i to get a new continued fraction $1/a_n + 1/a_{n-1} + \cdots + 1/a_1$. Compute the value p/q of this continued fraction, and also compute the strip of triangles for this fraction p/q.

<u>*Solution*</u>: $1/3 + 1/2 + 1/5 + 1/2 = 24/83$ as the strip of triangles illustrates:



Notice that the denominator 83 is the same as in the previous fraction $38/83$, and the product of the numerators is $38 \cdot 24 = 912$ which is congruent to -1 modulo 83. The general theory said the product of the numerators had to be congruent to ± 1, with a plus sign if the continued fraction has an odd number of terms and a minus sign if the continued fraction has an even number of terms.

5. Let p_n/q_n be the value of the continued fraction $1/a_1 + 1/a_2 + \cdots + 1/a_n$ where each of the n terms a_i is equal to 2. For example, $p_1/q_1 = 1/2$ and $p_2/q_2 = 1/2 + 1/2 = 2/5$.

(a) Find equations expressing p_n and q_n in terms of p_{n-1} and q_{n-1}, and use these to write down the values of p_n/q_n for $n = 1, 2, 3, 4, 5, 6, 7$.

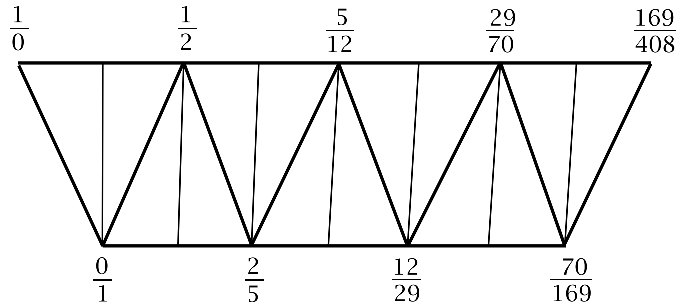
<u>*Solution*</u>: We have $p_n/q_n = 1/2 + 1/2 + \cdots + 1/2$ with n terms. This means that $\dfrac{p_n}{q_n} = \dfrac{1}{2 + \frac{p_{n-1}}{q_{n-1}}} = \dfrac{q_{n-1}}{2q_{n-1} + p_{n-1}}$ so we have $p_n = q_{n-1}$ and $q_n = 2q_{n-1} + p_{n-1}$.

[A technical point: If we use the formulas $p_n = q_{n-1}$ and $q_n = 2q_{n-1} + p_{n-1}$ to define each fraction p_n/q_n in terms of the preceding fraction p_{n-1}/q_{n-1}, then we always get fractions that are in lowest terms. The reason for this is that if the two numbers $p_n = q_{n-1}$ and $q_n = 2q_{n-1} + p_{n-1}$ had a common divisor $d > 1$, then d would also have to divide p_{n-1} since it divides both q_{n-1} and $2q_{n-1} + p_{n-1}$. Thus the preceding fraction p_{n-1}/q_{n-1} would not be in lowest terms. Repeating this reasoning, we would conclude that all the preceding fractions $p_1/q_1, p_2/q_2, \cdots$ were not in lowest terms, but $p_1/q_1 = 1/2$ which is in lowest terms. Hence p_n/q_n had to be in lowest terms.]

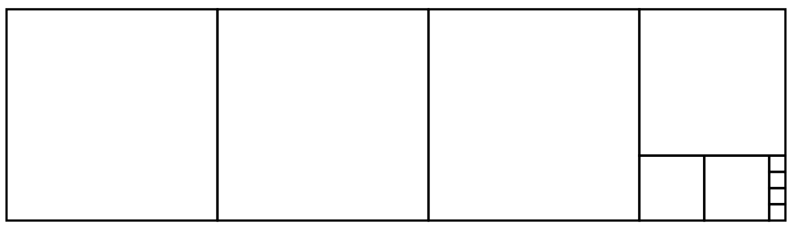
The formulas $p_n = q_{n-1}$ and $q_n = 2q_{n-1} + p_{n-1}$ can be used to compute each p_n/q_n from the preceding one. Doing this for the first seven cases gives $1/2$, $2/5$, $5/12$, $12/29$, $29/70$, $70/169$, $169/408$.

(b) Compute the strip of triangles for p_7/q_7.

Solution:



6. (a) A rectangle whose sides have lengths 13 and 48 can be partitioned into squares in the following way:



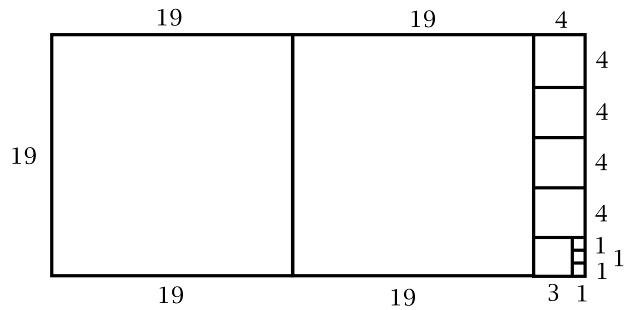
Determine the lengths of the sides of all the squares, and relate the numbers of squares of each size to the continued fraction for $13/48$.

Solution: The rectangle is 13 units high so the the large squares are 13×13. There are three of them totaling 39 units wide, so that leaves 9 units for the square in the upper right, making this a 9×9 square. The two larger squares below this square are then 4×4 and the smallest squares are 1×1.

The continued fraction for $13/48$ is ${}^{1}\!\nearrow_{3} + {}^{1}\!\nearrow_{1} + {}^{1}\!\nearrow_{2} + {}^{1}\!\nearrow_{4}$. The terms $3, 1, 2, 4$ are equal to the number of squares of each successively smaller size. This is not an accident since when you compute the continued fraction you get $48/13 = 3 + 9/13$ and the region to the right of the three big squares is a 9×13 rectangle. Next you compute $13/9 = 1 + 4/9$ and after the 9×9 square is removed from the 9×13 rectangle, what remains is a 4×9 rectangle. Finally, $9/4 = 2 + 1/4$ and after the two 4×4 squares are removed from the 4×9 rectangle, what remains is a 1×4 rectangle.

(b) Draw the analogous figure decomposing a rectangle of sides 19 and 42 into squares, and relate this to the continued fraction for $19/42$.

Solution:



Counting the numbers of squares of each size, we see that the continued fraction for $19/42$ is ${}^{1}\!\nearrow_{2} + {}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{1} + {}^{1}\!\nearrow_{3}$.