1. This exercise is intended to illustrate the proof of the Theorem on page 15 of Chapter 1 in the concrete case of the continued fraction ${}^{1}\!\nearrow_{2} + {}^{1}\!\nearrow_{3} + {}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{5}$.

(a) Write down the product $A_1A_2A_3A_4 = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_4 \end{pmatrix}$ associated to ${}^{1}\!\nearrow_{2} + {}^{1}\!\nearrow_{3} + {}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{5}$.

(b) Compute the four matrices A_1, A_1A_2, $A_1A_2A_3$, $A_1A_2A_3A_4$ and relate these to the edges of the zigzag path in the strip of triangles for ${}^{1}\!\nearrow_{2} + {}^{1}\!\nearrow_{3} + {}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{5}$.

(c) Compute the four matrices A_4, A_3A_4, $A_2A_3A_4$, $A_1A_2A_3A_4$ and relate these to the successive fractions that one gets when one computes the value of ${}^{1}\!\nearrow_{2} + {}^{1}\!\nearrow_{3} + {}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{5}$, namely ${}^{1}\!\nearrow_{5}$, ${}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{5}$, ${}^{1}\!\nearrow_{3} + {}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{5}$, and ${}^{1}\!\nearrow_{2} + {}^{1}\!\nearrow_{3} + {}^{1}\!\nearrow_{4} + {}^{1}\!\nearrow_{5}$.

2. (a) Find all integer solutions of the equations $40x + 89y = 1$ and $40x + 89y = 5$.

(b) Find another equation $ax + by = 1$ that has an integer solution in common with $40x + 89y = 1$. [Hint: use the Farey diagram.]

3. There is a close connection between the Diophantine equation $ax+by = n$ and the congruence $ax \equiv n \bmod b$, where the symbol $\equiv$ means "is congruent to". Namely, if one has a solution (x, y) to $ax + by = n$ then $ax \equiv n \bmod b$, and conversely, if one has a number x such that $ax \equiv n \bmod b$ then this means that $ax - n$ is a multiple of b, say k times b, so $ax - n = kb$ or equivalently $ax - kb = n$ so one has a solution of $ax + by = n$ with $y = -k$.

Using this viewpoint, solve the congruences $31x \equiv 1 \bmod 71$ and $31x \equiv 10 \bmod 71$. Are the solutions unique mod 71, i.e., unique up to adding multiples of 71?