1. Compute the values of the following infinite continued fractions:

(a) $\overline{{}^1\!/_4}$

<u>*Solution*</u>: Let $x = {}^1\!/_4 + {}^1\!/_4 + \cdots$. Then $1/x = 4 + {}^1\!/_4 + {}^1\!/_4 + \cdots = 4 + x$. This gives the quadratic equation $1 = 4x + x^2$, or $x^2 + 4x - 1 = 0$, with solutions $x = (-4 \pm \sqrt{20})/2 = -2 \pm \sqrt{5}$. We want the positive root $-2 + \sqrt{5}$.

(b) $\overline{{}^1\!/_k}$ for an arbitrary positive integer k.

<u>*Solution*</u>: Let $x = {}^1\!/_k + {}^1\!/_k + \cdots$. Then $1/x = k + {}^1\!/_k + {}^1\!/_k + \cdots = k + x$. Thus we have $1 = kx + x^2$, so $x^2 + kx - 1 = 0$ with roots $x = (-k \pm \sqrt{k^2 + 4})/2$ and we want the positive root $(-k + \sqrt{k^2 + 4})/2$.

(c) $\overline{{}^1\!/_2 + {}^1\!/_3}$ and ${}^1\!/_1 + \overline{{}^1\!/_2 + {}^1\!/_3}$

<u>*Solution*</u>: Let $x = \overline{{}^1\!/_2 + {}^1\!/_3}$. Then $1/x - 2 = {}^1\!/_3 + x$. After rewriting the left side of this equation as $(1 - 2x)/x$, we get the equation $x/(1 - 2x) = 3 + x$. This simplifies to $2x^2 + 6x - 3 = 0$ with positive root $x = \overline{{}^1\!/_2 + {}^1\!/_3} = (-3 + \sqrt{15})/2$.

To compute ${}^1\!/_1 + \overline{{}^1\!/_2 + {}^1\!/_3}$ we take the previous answer, add 1, then invert, to get $2/(-1 + \sqrt{15}) = (1 + \sqrt{15})/7$.

(d) $\overline{{}^1\!/_1 + {}^1\!/_2 + {}^1\!/_1 + {}^1\!/_6}$ and ${}^1\!/_1 + {}^1\!/_4 + \overline{{}^1\!/_1 + {}^1\!/_2 + {}^1\!/_1 + {}^1\!/_6}$

<u>*Solution*</u>: We set $x = \overline{{}^1\!/_1 + {}^1\!/_2 + {}^1\!/_1 + {}^1\!/_6}$. Then $1/x = 1 + {}^1\!/_2 + {}^1\!/_1 + {}^1\!/_6 + x$ and $1/x - 1 = (1 - x)/x = {}^1\!/_2 + {}^1\!/_1 + {}^1\!/_6 + x$, so $x/(1 - x) = 2 + {}^1\!/_1 + {}^1\!/_6 + x$. Subtracting 2 from both sides and inverting leads to $(1 - x)/(3x - 2) = 1 + {}^1\!/_6 + x$. Then subtract 1 from both sides and invert to get $(3x - 2)/(4 - 3x) = 6 + x$. This simplifies to $x^2 + 6x - 5 = 0$, with positive root $x = \overline{{}^1\!/_1 + {}^1\!/_2 + {}^1\!/_1 + {}^1\!/_6} = -3 + \sqrt{14}$.

To go from this to ${}^1\!/_1 + {}^1\!/_4 + \overline{{}^1\!/_1 + {}^1\!/_2 + {}^1\!/_1 + {}^1\!/_6}$ we add 4 and invert, then add 1 and invert. This produces the answer of $(12 - \sqrt{14})/10$.

(e) $\overline{{}^1\!/_2 + {}^1\!/_3 + {}^1\!/_5}$

<u>*Solution*</u>: Set $x = \overline{{}^1\!/_2 + {}^1\!/_3 + {}^1\!/_5}$, so $1/x = 2 + {}^1\!/_3 + {}^1\!/_5 + x$. rewrite this equation as $1/x - 2 = (1 - 2x)/x = {}^1\!/_3 + {}^1\!/_5 + x$, and inverting this gives $x/(1 - 2x) = 3 + {}^1\!/_5 + x$. Then $x/(1 - 2x) - 3 = (7x - 3)/(1 - 2x) = {}^1\!/_5 + x$ and inverting this gives $(1 - 2x)/(7x - 3) = 5 + x$ which simplifies to the quadratic equation $7x^2 + 34x - 16 = 0$. The positive root of this equation is $(-17 + \sqrt{401})/7$. This calculation shows that a fairly simple-looking continued fraction like $\overline{{}^1\!/_2 + {}^1\!/_3 + {}^1\!/_5}$

can have a fairly complicated value.

2. Compute the continued fractions for $\sqrt{5}$ and $\sqrt{23}$.

Solution: Since $\sqrt{5}$ is between 2 and 3 we have

$$\sqrt{5} = 2 + (\sqrt{5} - 2) \quad \text{and} \quad \frac{1}{\sqrt{5} - 2} = \sqrt{5} + 2$$
$$\sqrt{5} + 2 = 4 + (\sqrt{5} - 2)$$

This is the same remainder as in the first line, so we conclude that $\sqrt{5} = 2 + \overline{1/4}$.

For $\sqrt{23}$ which is between 4 and 5 we have

$$\begin{aligned}
\sqrt{23} &= 4 + (\sqrt{23} - 4) \quad &&\text{and} \quad \frac{1}{\sqrt{23} - 4} = \frac{\sqrt{23} + 4}{7} \\
\frac{\sqrt{23} + 4}{7} &= 1 + \frac{\sqrt{23} - 3}{7} \quad &&\text{and} \quad \frac{7}{\sqrt{23} - 3} = \frac{7(\sqrt{23} + 3)}{14} = \frac{\sqrt{23} + 3}{2} \\
\frac{\sqrt{23} + 3}{2} &= 3 + \frac{\sqrt{23} - 3}{2} \quad &&\text{and} \quad \frac{2}{\sqrt{23} - 3} = \frac{2(\sqrt{23} + 3)}{14} = \frac{\sqrt{23} + 3}{7} \\
\frac{\sqrt{23} + 3}{7} &= 1 + \frac{\sqrt{23} - 4}{7} \quad &&\text{and} \quad \frac{7}{\sqrt{23} - 4} = \frac{7(\sqrt{23} + 4)}{7} = \sqrt{23} + 4 \\
\sqrt{23} + 4 &= 8 + (\sqrt{23} - 4)
\end{aligned}$$

Finally we are back to the original remainder, so $\sqrt{23} = 4 + \overline{1/1 + 1/3 + 1/1 + 1/8}$.

3. Compute the continued fractions for $\sqrt{n^2 + 1}$ and $\sqrt{n^2 + n}$ where n is an arbitrary positive integer.

Solution: $\sqrt{n^2 + 1}$ is between n and $n + 1$ so we have

$$\sqrt{n^2 + 1} = n + (\sqrt{n^2 + 1} - n) \quad \text{and} \quad \frac{1}{\sqrt{n^2 + 1} - n} = \sqrt{n^2 + 1} + n$$
$$\sqrt{n^2 + 1} + n = 2n + (\sqrt{n^2 + 1} - n)$$

So we have a repeated remainder and $\sqrt{n^2 + 1} = n + \overline{1/2n}$.

For $\sqrt{n^2 + n}$, this is also between n and $n + 1$ and we have

$$\begin{aligned}
\sqrt{n^2 + n} &= n + (\sqrt{n^2 + n} - n) \quad &&\text{and} \quad \frac{1}{\sqrt{n^2 + n} - n} = \frac{\sqrt{n^2 + n} + n}{n} \\
\frac{\sqrt{n^2 + n} + n}{n} &= 2 + \frac{\sqrt{n^2 + n} - n}{n} \quad &&\text{and} \quad \frac{n}{\sqrt{n^2 + n} - n} = \sqrt{n^2 + n} + n \\
\sqrt{n^2 + n} + n &= 2n + (\sqrt{n^2 + n} - n)
\end{aligned}$$

which brings us to a repeated remainder, so $\sqrt{n^2 + n} = n + \overline{1/2 + 1/2n}$.