1. Find a formula for the linear fractional transformation that rotates the triangle $\langle 0/1, 1/2, 1/1\rangle$ to $\langle 1/1, 0/1, 1/2\rangle$.

Solution: A rotation preserves orientation hence has determinant $+1$, so we want to use only matrices of determinant $+1$ if possible. A transformation of determinant $+1$ that takes the edge $\langle 0/1, 1/2\rangle$ to $\langle 1/1, 0/1\rangle$ is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

Here we inserted a minus sign in the first column of the second matrix in order to make its determinant $+1$. One can check that the final matrix takes the third vertex of $\langle 0/1, 1/2, 1/1\rangle$ to the third vertex of $\langle 1/1, 0/1, 1/2\rangle$, but this check isn't really necessary since a transformation of determinant $+1$ is uniquely determined by where it sends an edge. (An alternative approach would be to change the sign of one column of the first matrix rather than the second matrix, so that the final matrix would again have determinant $+1$.)

2. Find the linear fractional transformation that reflects the Farey diagram across the edge $\langle 1/2, 1/3\rangle$ (so in particular, the transformation takes $1/2$ to $1/2$ and $1/3$ to $1/3$).

Solution: We want to take $\langle 1/2, 1/3\rangle$ to $\langle 1/2, 1/3\rangle$ by a transformation of determinant -1 since reflections reverse orientation. A matrix that does this is

$$\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ -12 & 5 \end{pmatrix}$$

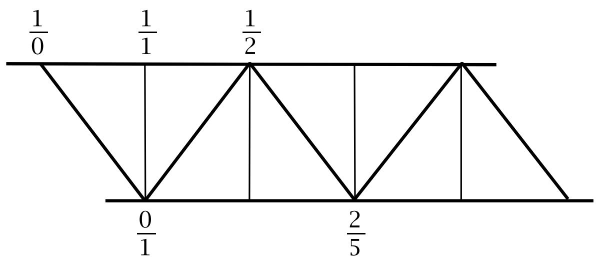
where we changed the sign of one column of the first matrix in order to make the determinant -1.

3. Find a formula for the linear fractional transformation that reflects the upper half-plane version of the Farey diagram across the vertical line $x = 3/2$.

Solution: This reflection takes the vertical edge $\langle 1/0, 0/1\rangle$ in the upper halfplane model of the Farey diagram to the vertical edge $\langle 1/0, 3/1\rangle$. A matrix that does this is $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ but this has determinant $+1$ whereas we want a matrix of determinant -1 for a reflection. All we need to do is change the sign of one column to get a matrix that works, $\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$.

4. Find an infinite periodic strip of triangles in the Farey diagram such that the transformation $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ is a glide-reflection along this strip and the transformation $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ is a translation along this strip.

<u>*Solution*</u>: The transformation $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ takes the edge $\langle 1/0, 0/1 \rangle$ to $\langle 0/1, 1/2 \rangle$, so for a start we can draw the part of the Farey diagram containing these two edges. This gives the two triangles on the left in the following figure:



Now if we continue drawing triangles as in the rest of the figure, with two triangles in each fan, we get an infinite periodic strip such that $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ is a glide-reflection along this strip, carrying each fan to the next fan. The translation that carries each fan to the second fan to the right of it has matrix $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$.

5. Let T be an element of $LF(\mathbb{Z})$ with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that the composition $T\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}T^{-1}$ is the reflection across the edge $\langle a/c, b/d \rangle = T(\langle 1/0, 0/1 \rangle)$.

<u>*Solution*</u>: The reflection across the edge $\langle a/c, b/d \rangle$ is characterized as the unique element of $LF(\mathbb{Z})$ that fixes the edge $\langle a/c, b/d \rangle$ and reverses orientation. To see that $T\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}T^{-1}$ fixes $\langle a/c, b/d \rangle$, note that T^{-1} takes $\langle a/c, b/d \rangle$ to $\langle 1/0, 0/1 \rangle$, then $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ fixes $\langle 1/0, 0/1 \rangle$, then T takes $\langle 1/0, 0/1 \rangle$ back to $\langle a/c, b/d \rangle$, so the net result is that $T\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}T^{-1}$ fixes $\langle a/c, b/d \rangle$. To see that $T\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}T^{-1}$ reverses orientation we can compute its determinant. Note first the general fact that $\det(ABA^{-1}) = \det(B)$ since

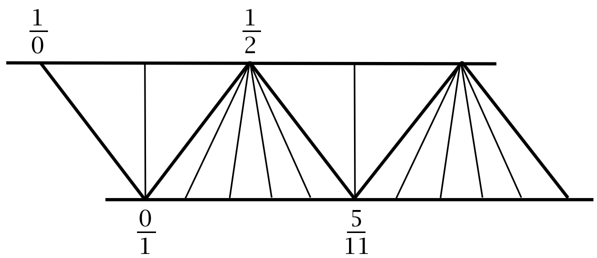
$$\begin{aligned} \det(ABA^{-1}) &= \det(A)\det(B)\det(A^{-1}) = \det(A)\det(A^{-1})\det(B) \\ &= \det(AA^{-1})\det(B) = \det(I)\det(B) = \det(B) \end{aligned}$$

Applying this to $T\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}T^{-1}$, we see that it has the same determinant as $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so the determinant is -1.

One can also give a more geometric argument: T^{-1} takes the vertices $v = T(1/1)$ and $w = T(-1/1)$, which are the third vertices of the two triangles on either side of the edge $\langle a/c, b/d \rangle$, to the vertices $1/1$ and $-1/1$. Then $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ switches these two vertices $1/1$ and $-1/1$, and finally T takes these switched vertices back to w and v, in the opposite order. So the net result is that $T\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}T^{-1}$ switches v and w, which means that it is a reflection.

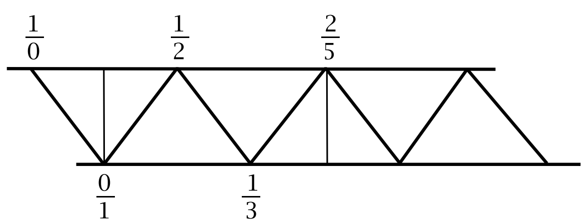
For each of the remaining problems, compute the value of the given periodic or eventually periodic continued fraction by first drawing the associated infinite strip of triangles, then finding a linear fractional transformation T in $LF(\mathbb{Z})$ that gives the periodicity in the strip, then solving $T(z) = z$.

6. $\overline{{}^1\!\diagup\!_2 + {}^1\!\diagup\!_5}$



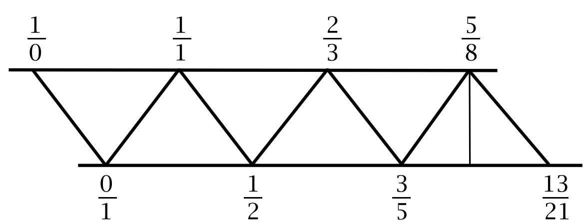
The periodicity transformation is the translation sending $\langle 1/0, 0/1\rangle$ to $\langle 1/2, 5/11\rangle$, with matrix $\begin{pmatrix} 1 & 5 \\ 2 & 11 \end{pmatrix}$ of determinant 1. The equation $T(z) = z$ is $\frac{z+5}{2z+11} = z$, or $z + 5 = 2z^2 + 11z$ which simplifies to $2z^2 + 10z - 5 = 0$, with roots $(-5 \pm \sqrt{35})/2$. The value of the continued fraction is the positive root $(-5 + \sqrt{35})/2$.

7. $\overline{{}^1\!\diagup\!_2 + {}^1\!\diagup\!_1 + {}^1\!\diagup\!_1}$



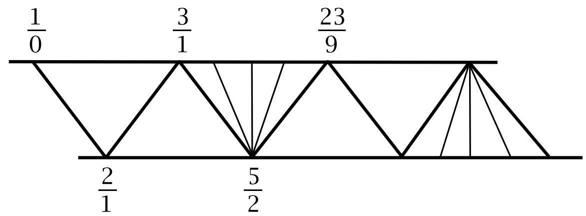
The transformation is a glide-reflection with matrix $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ of determinant -1. This gives the equation $z + 2 = 3z^2 + 5z$, so $3z^2 + 4z - 2 = 0$ and the positive root is $(-2 + \sqrt{10})/3$.

8. $\overline{{}^1\!\diagup\!_1 + {}^1\!\diagup\!_1 + {}^1\!\diagup\!_1 + {}^1\!\diagup\!_1 + {}^1\!\diagup\!_1 + {}^1\!\diagup\!_2}$



The periodicity is the translation $\begin{pmatrix} 5 & 13 \\ 8 & 21 \end{pmatrix}$ with determinant 1 so we get $5z + 13 = 8z^2 + 21z$, simplifying to $8z^2 + 16z - 13 = 0$ with positive root $(-4 + \sqrt{42})/4$.

9. $2 + \overline{{}^1\!/_1 + {}^1\!/_1 + {}^1\!/_4}$

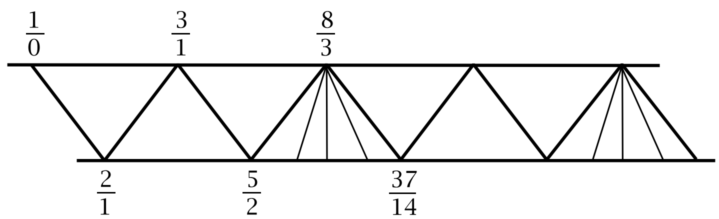


The periodicity transformation is a glide-reflection taking $\langle 1/0, 2/1\rangle$ to $\langle 5/2, 23/9\rangle$ with matrix

$$\begin{pmatrix} 5 & 23 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 23 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 13 \\ 2 & 5 \end{pmatrix}$$

having determinant -1. The equation is $5z+13 = 2z^2+5z$ so $z = \sqrt{13/2}$, the positive root.

10. $2 + \overline{{}^1\!/_1 + {}^1\!/_1 + {}^1\!/_1 + {}^1\!/_4}$

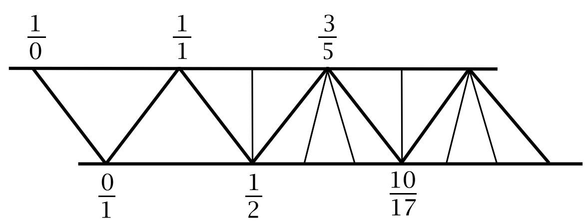


The periodicity transformation is a translation taking $\langle 1/0, 2/1\rangle$ to $\langle 8/3, 37/14\rangle$ with matrix

$$\begin{pmatrix} 8 & 37 \\ 3 & 14 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 8 & 37 \\ 3 & 14 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 21 \\ 3 & 8 \end{pmatrix}$$

having determinant 1. The equation is $8z + 21 = 3z^2 + 8z$ so $z = \sqrt{7}$, the positive root.

11. ${}^1\!/_1 + {}^1\!/_1 + \overline{{}^1\!/_2 + {}^1\!/_3}$



The periodicity transformation is a translation taking $\langle 1/1, 1/2\rangle$ to $\langle 3/5, 10/17\rangle$ with matrix

$$\begin{pmatrix} 3 & 10 \\ 5 & 17 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 10 \\ 5 & 17 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -7 & 12 \end{pmatrix}$$

having determinant 1. The equation is $-4z + 7 = -7z^2 + 12z$ simplifying to $7z^2 - 16z + 7 = 0$ so $z = (8 \pm \sqrt{15})/7$. Both roots are positive and we want the smaller one $(8 - \sqrt{15})/7$ since we are moving to the right in the infinite periodic strip in the upper half of the Farey diagram. (The other root is bigger than 1 but the value of the given continued fraction is less than 1 since its integer part is $a_0 = 0$.)