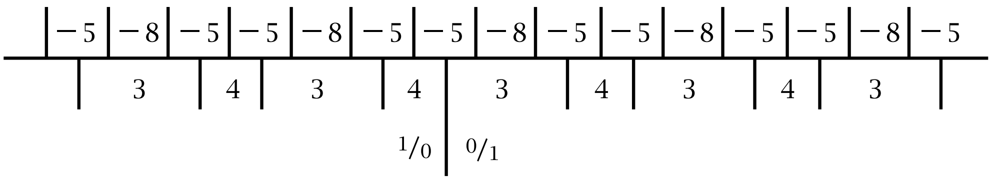
1. Use a quadratic form to compute continued fractions for the following pairs of numbers:

(a) $(3+\sqrt{6})/2$ and $(3-\sqrt{6})/2$

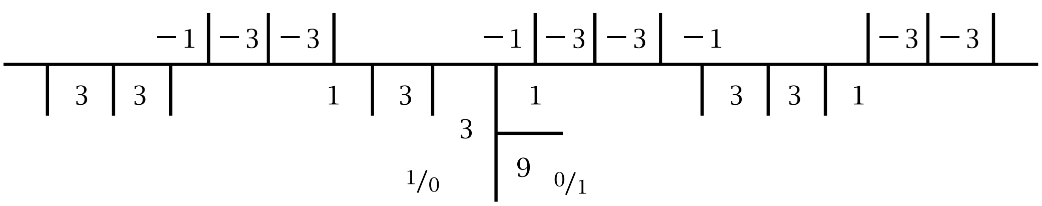
<u>*Solution*</u>: First we find a quadratic equation having these two numbers as its roots. For roots α and $\overline{\alpha}$ an equation is $x^2-(\alpha+\overline{\alpha})x+\alpha\overline{\alpha}=0$. In the present case this gives $x^2-3x+\frac{3}{4}=0$ or $4x^2-12x+3=0$. The corresponding quadratic form is $4x^2-12xy+3y^2$. Here is the separator line for this form:



Both roots are positive. The smaller one $(3-\sqrt{6})/2$ is toward the right (since we're in the upper half of the Farey diagram). The sequence of side roads, starting at the $\langle 1/0, 0/1\rangle$ edge and going off to the right, is $L^3\overline{RLRL^2}$. This gives the continued fraction $\frac{1}{3}+\overline{\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}}$. For the larger root $(3+\sqrt{6})/2$ we go off to the left, with side roads $R^2\overline{LR^2LR}$, so the continued fraction is $2+\overline{\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{1}}$.

(b) $(11+\sqrt{13})/6$ and $(11-\sqrt{13})/6$

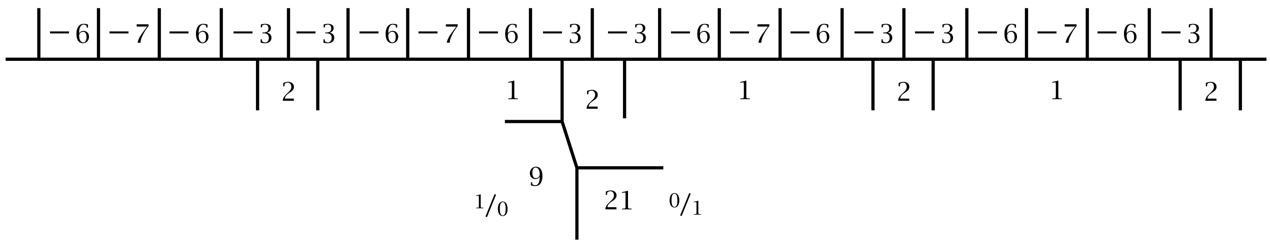
<u>*Solution*</u>: The equation having these roots is $3x^2-11x+9=0$ so the quadratic form is $3x^2-11xy+9y^2$. Its separator line is:



Both roots are positive. For the smaller root $(11-\sqrt{13})/6$ off to the right in the topograph the side road sequence is $RL^4\overline{R^3L^3}$ so the continued fraction is $1+\frac{1}{4}+\overline{\frac{1}{3}}$. For the larger root $(11+\sqrt{13})/6$ the sequence is $R^2L^2\overline{R^3L^3}$ so the continued fraction is $2+\frac{1}{2}+\overline{\frac{1}{3}}$.

(c) $(14+\sqrt{7})/9$ and $(14-\sqrt{7})/9$

<u>*Solution*</u>: The equation with these roots is $9x^2-28x+21=0$ so the quadratic form is $9x^2-28xy+21y^2$ with the following separator line:

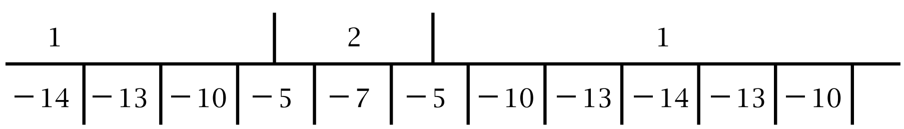


For the smaller root $(14-\sqrt{7})/9$ we have $RL^3\overline{RL^4RL}$ with continued fraction $1+{}^1\!\nearrow_3+\overline{{}^1\!\nearrow_1+{}^1\!\nearrow_4+{}^1\!\nearrow_1+{}^1\!\nearrow_1}$. For the larger root $(14+\sqrt{7})/9$ we have $RLR^5\overline{LRLR^4}$ with the continued fraction $1+{}^1\!\nearrow_1+{}^1\!\nearrow_5+\overline{{}^1\!\nearrow_1+{}^1\!\nearrow_1+{}^1\!\nearrow_1+{}^1\!\nearrow_4}$.

2. For the quadratic form x^2-14y^2 do the following things:

(a) Draw the separator line in the topograph and compute the continued fraction for $\sqrt{14}$.

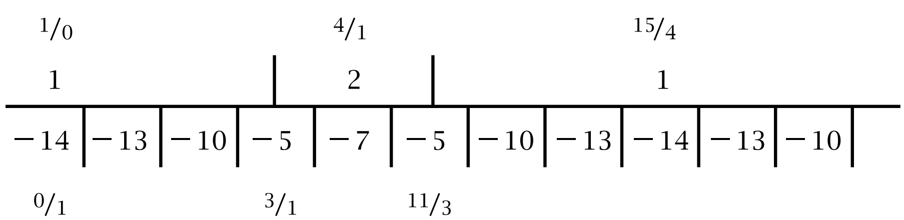
<u>*Solution*</u>:



From this we read off $\sqrt{14}=3+\overline{{}^1\!\nearrow_1+{}^1\!\nearrow_2+{}^1\!\nearrow_1+{}^1\!\nearrow_6}$.

(b) Find the smallest positive integer solutions of $x^2-14y^2=1$ and $x^2-14y^2=-1$, if these equations have integer solutions.

<u>*Solution*</u>: There are two ways to do this. Either we can fill in a few of the x/y labels along the separator line:



and from this we get the solution $(x,y)=(15,4)$ for the equation $x^2-14y^2=1$, or we can compute the value of the finite continued fraction obtained from the continued fraction for $\sqrt{14}$ by stopping before the term ${}^1\!\nearrow_6$. This gives $3+{}^1\!\nearrow_1+{}^1\!\nearrow_2+{}^1\!\nearrow_1=15/4$. The equation $x^2-14y^2=-1$ has no integer solutions since -1 does not appear along the separator line, and as we move away from the separator line the values of the form only become larger (in absolute value).

(c) Find the linear fractional transformation that gives the periodicity translation along the separator line and use this to find a second positive solution of $x^2 - 14y^2 = 1$.

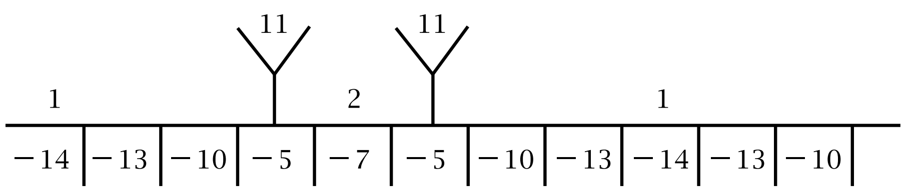
<u>*Solution*</u>: This transformation has matrix $\begin{pmatrix} p & dq \\ q & p \end{pmatrix}$ where $(p,q) = (15,4)$ from part (b). This gives $\begin{pmatrix} 15 & 56 \\ 4 & 15 \end{pmatrix}$. To get the next solution of $x^2 - 14y^2 = 1$ after $(15,4)$ we apply this transformation to the first solution:

$$\begin{pmatrix} 15 & 56 \\ 4 & 15 \end{pmatrix}\begin{pmatrix} 15 \\ 4 \end{pmatrix} = \begin{pmatrix} 449 \\ 120 \end{pmatrix}$$

So the next solution is $(x,y) = (449,120)$.

(d) Determine the integers n with $|n| \le 12$ such that the equation $x^2 - 14y^2 = n$ has an integer solution. (Don't forget the possibility that there could be solutions (x,y) that aren't primitive.)

<u>*Solution*</u>: For a start we need to find all values n in the topograph with $|n| \le 12$. Besides the values along the separator line, there is one more value not on the separator line:

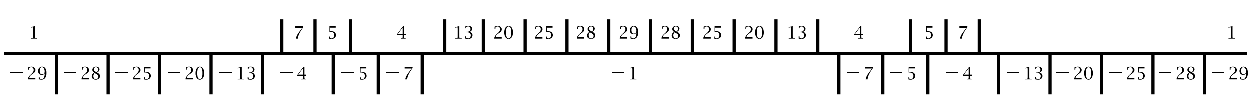


Thus we have the values $-10, -7, -5, 1, 2, 11$. We also have to multiply these numbers by squares. This gives three more values for n: 4, 8, and 9.

3. For the quadratic form $x^2 - 29y^2$ do the following things:

(a) Draw the separator line and compute the continued fraction for $\sqrt{29}$.

<u>*Solution*</u>: Here is the separator line:



From this we get $\sqrt{29} = 5 + \overline{\frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{10}}$.

(b) Find the smallest positive integer solution of $x^2 - 29y^2 = -1$.

<u>*Solution*</u>: From the convergent $5 + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} = 70/13$ we get $(x,y) = (70,13)$.

(c) Find a glide-reflection symmetry of the separator line and use this to find the smallest positive integer solution of $x^2 - 29y^2 = 1$.

<u>*Solution*</u>: The glide-reflection has matrix $\begin{pmatrix} p & dq \\ q & p \end{pmatrix} = \begin{pmatrix} 70 & 377 \\ 13 & 70 \end{pmatrix}$. We apply this to $70/13$ to get

$$\begin{pmatrix} 70 & 377 \\ 13 & 70 \end{pmatrix} \begin{pmatrix} 70 \\ 13 \end{pmatrix} = \begin{pmatrix} 9801 \\ 1820 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

4. Compute the periodic separator line for the form $x^2 - 43y^2$ and use this to find the continued fraction for $\sqrt{43}$.

<u>*Solution*</u>: The separator line is:

1	6	13	14	9	17	21	21	17	9	14	13	6	1

−43	−42	−39	−34	−27	−18	−7	−3	−2	−3	−7	−18	−27	−34	−39	−42	−43

From this we get $\sqrt{43} = 6 + \overline{\tfrac{1}{1} + \tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{1} + \tfrac{1}{5} + \tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{1} + \tfrac{1}{1} + \tfrac{1}{12}}$.