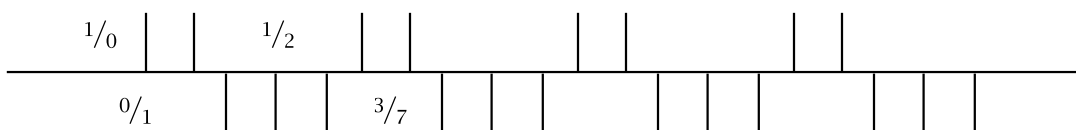


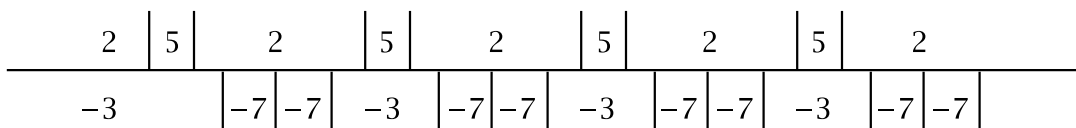
1. Find a hyperbolic quadratic form whose periodic separator line has the following pattern:



Solution: We can insert x/y labels in the following way



and then the periodicity transformation has matrix $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$. This transformation is $T(z) = (z+3)/(2z+7)$. We would find its fixed points by solving $(z+3)/(2z+7) = z$, which simplifies to $z+3 = 2z^2+7z$ or $2z^2+6z-3 = 0$. We don't actually have to solve this, however. Instead we just take the associated quadratic form $2x^2 + 6xy - 3y^2$. The separator line for this form is:

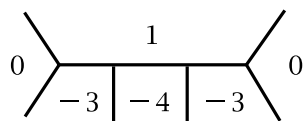


2. In this problem we consider only quadratic forms $Q(x, y) = ax^2 + cy^2$ with no xy term, for the sake of simplicity.

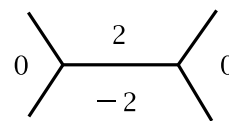
(a) Find two 0-hyperbolic forms $ax^2 + cy^2$ that have the same discriminant but take on different sets of values. Draw enough of the topographs of the two forms to make it apparent that they do not have exactly the same sets of values. (Remember that the topograph only shows the values $Q(x, y)$ for primitive pairs (x, y) .)

Solution: For a form $ax^2 + cy^2$ the discriminant is $\Delta = -4ac$. Different choices of a and c with the same product ac then give forms with the same discriminant. For example, consider $x^2 - 4y^2$ and $2x^2 - 2y^2$. These both have discriminant 16, a positive square, so they are 0-hyperbolic. The form $x^2 - 4y^2$ takes the value 1 but the form $2x^2 - 2y^2$ obviously takes only even values, so the two forms do not take the same sets of values. This can also be checked by looking at the finite separator line in the topographs:

$$\underline{x^2 - 4y^2}$$



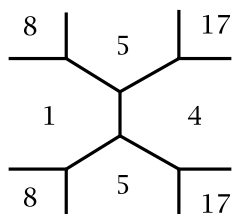
$$\underline{2x^2 - 2y^2}$$



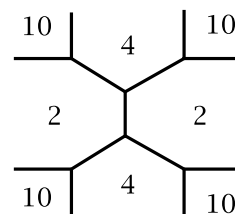
(b) Do the same thing with two elliptic forms that take on positive values. Include the source vertex or source edge in the topographs.

Solution: Following the same idea, consider $x^2 + 4y^2$ and $2x^2 + 2y^2$, which both have discriminant -16 so they are elliptic. The first form takes the value 1 while the second only takes even values, so they have different sets of values. Their topographs have source edges:

$$\underline{x^2 + 4y^2}$$



$$\underline{2x^2 + 2y^2}$$

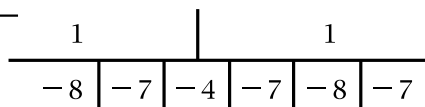


(c) Do the same thing with two hyperbolic forms, drawing their separator lines.

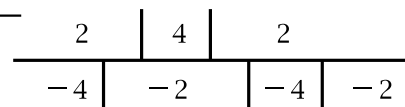
Solution: This time consider $x^2 - 8y^2$ and $2x^2 - 4y^2$, forms of discriminant 32 which is positive and not a square, so they are hyperbolic. They have different sets of values since the first form takes the value 1 while the second only takes even values.

Here are their periodic separator lines:

$$\underline{x^2 - 8y^2}$$



$$\underline{2x^2 - 4y^2}$$



square.

Solution: The form $Q(x, y) = (Ax + By)(Cx + Dy)$ takes the value 0 at $(x, y) = (B, -A)$. This pair $(B, -A)$ cannot be $(0, 0)$ since if it were, then from the formula $Q(x, y) = (Ax + By)(Cx + Dy)$ the form Q would be identically zero, with $Q(x, y) = 0$ for all (x, y) , a case which we always exclude implicitly.

To compute the discriminant of the form $(Ax + By)(Cx + Dy)$ we first multiply this out to get $ACx^2 + (AD + BC)xy + BDy^2$, and then the discriminant is

$$(AD + BC)^2 - 4ACBD = (AD - BC)^2$$

which is a square.

(b) Find a 0-hyperbolic form $Q(x, y)$ such that $Q(1, 5) = 0$ and $Q(7, 2) = 0$ and draw a portion of the topograph of Q that includes the two regions where $Q = 0$.

Solution: As in the solution in part (a) we see that a linear factor $5x - y$ equals 0 at $(1, 5)$ and a linear factor $2x - 7y$ equals 0 at $(7, 2)$, so we take the quadratic form $Q(x, y) = (5x - y)(2x - 7y) = 10x^2 - 37xy + 7y^2$. The separator line connecting the two regions where $Q(x, y) = 0$ is shown below:

