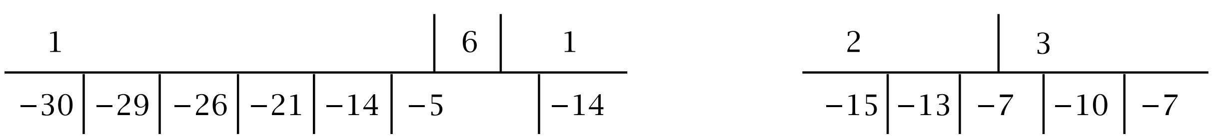
1. Determine the number of equivalence classes of quadratic forms of discriminant $\Delta = 120$ and list one form from each equivalence class.

<u>*Solution*</u>: We use the relationship $\Delta = h^2 + 4pq$ which holds for each edge of the separator line, where h is the label on the edge and the adjacent regions are labeled p and $-q$, with $p > 0$ and $q > 0$. We also know that h has the same parity as Δ, because of the formula $\Delta = h^2 + 4pq$. Now we make a table of all the possibilities for h, p, and q when $\Delta = 120$:

h	pq	(p,q)
0	30	$(1,30),\ (2,15),\ (3,10),\ (5,6),\ (6,5),\ (10,3),\ (15,2),\ (30,1)$
2	29	$(1,29),\ (29,1)$
4	26	$(1,26),\ (2,13),\ (13,2),\ (26,1)$
6	21	$(1,21),\ (3,7),\ (7,3),\ (21,1)$
8	14	$(1,14),\ (2,7),\ (7,2),\ (14,1)$
10	5	$(1,5),\ (5,1)$

(Notice incidentally that the values for pq decrease regularly, by $1,3,5,7,9$.) To find forms that realize all the possibilities in the table we start with the principal form $x^2 - 30y^2$, whose separator line is shown in the left half of the following figure:



This realizes all the pairs (p,q) in the table with p equal to 1 or 6. To realize $(2,15)$ we need a new form, so we take $2x^2 - 15y^2$ with topograph shown in the right half of the figure above. This takes care of all the pairs (p,q) with $p = 2$ or $p = 3$. To get all the remaining pairs (p,q) we use the negatives of these two forms, $-x^2 + 30y^2$ and $-2x^2 + 15y^2$. Thus all forms of discriminant 120 are equivalent to one of the four forms $x^2 - 30y^2$, $2x^2 - 15y^2$, $-x^2 + 30y^2$ and $-2x^2 + 15y^2$. No two of these four forms are equivalent to each other since their separator lines are different (the two separator lines shown and the negatives of these two separator lines).

2. Do the same thing for $\Delta = 61$.

<u>*Solution*</u>: This is similar, but now h is odd since Δ is odd. The table is:

h	pq	(p,q)
1	15	$(1,15),\ (3,5),\ (5,3),\ (15,1)$
3	13	$(1,13),\ (13,1)$
5	9	$(1,9),\ (3,3),\ (9,1)$
7	3	$(1,3),\ (3,1)$

The principal form of this discriminant is $x^2 + xy - 15y^2$ with separator line:

$$\begin{array}{c|c|c|c|c|c|c}
1 & 5 & 3 & 9 & 13 & 15 & 15 \\ \hline
\end{array}$$
$$\begin{array}{c|c|c|c|c|c}
-15 & -13 & -9 & -3 & -5 & -1
\end{array}$$

This realizes all the combinations (p,q) in the table, so every form of discriminant 61 is equivalent to $x^2 + xy - 15y^2$.

3. (a) Find the smallest positive nonsquare discriminant for which there is more than one equivalence class of forms of that discriminant.

Solution: Discriminants are congruent to 0 or 1 modulo 4 so the first few that are not squares are $5, 8, 12, 13, 17, 20$. For $\Delta = 5$ we have the table

h	pq	(p,q)
1	1	$(1,1)$

so there is only one equivalence class for $\Delta = 5$. For $\Delta = 8$ we have:

h	pq	(p,q)
0	2	$(1,2),\ (2,1)$
2	1	$(1,1)$

$$\begin{array}{c|c|c}
1 & 2 & 1 \\ \hline
\end{array}$$
$$\begin{array}{c|c|c}
-2 & -1 & -2
\end{array}$$

All these pairs (p,q) are realized by $x^2 - 2y^2$ with the topograph shown, so there is only one equivalence class for $\Delta = 8$. For $\Delta = 12$ we have:

h	pq	(p,q)
0	3	$(1,3),\ (3,1)$
2	2	$(1,2),\ (2,1)$

$$\begin{array}{c|c}
1 & 1 \\ \hline
\end{array}$$
$$\begin{array}{c|c|c}
-3 & -2 & -3
\end{array}$$

The topograph of $x^2 - 3y^2$ is shown, and it realizes only half the pairs (p,q). The rest are realized by $-x^2 + 3y^2$ so there are two equivalence classes for $\Delta = 12$. This is the first time this happens.

(b) Find the smallest positive nonsquare discriminant for which there are two inequivalent forms of that discriminant, neither of which is simply the negative of the other.

Solution: The case $\Delta = 12$ doesn't work here, so we continue further.

$\Delta = 13$:

h	pq	(p,q)
1	3	$(1,3),\ (3,1)$
3	1	$(1,1)$

$$\begin{array}{c|c|c|c} 1 & 3 & 3 & 1 \\ \hline \end{array}$$
$$\begin{array}{c|c|c} -3 & -1 & -3 \end{array}$$

All (p,q) are realized by $x^2 + xy - 3y^2$ so all forms with $\Delta = 13$ are equivalent.

$\Delta = 17$:

h	pq	(p,q)
1	4	$(1,4),\ (2,2),\ (4,1)$
3	2	$(1,2),\ (2,1)$

$$\begin{array}{c|c|c|c|c|c} 1 & 2 & 4 & 4 & 2 & 1 \\ \hline \end{array}$$
$$\begin{array}{c|c|c|c|c} -4 & -2 & -1 & -2 & -4 \end{array}$$

All (p,q) are realized by $x^2 + xy - 4y^2$ so all forms with $\Delta = 17$ are equivalent.

$\Delta = 20$:

h	pq	(p,q)
0	5	$(1,5),\ (5,1)$
2	4	$(1,4),\ (2,2),\ (4,1)$
4	1	$(1,1)$

$$\begin{array}{c|c|c|c|c} 1 & 4 & 5 & 4 & 1 \\ \hline \end{array}$$
$$\begin{array}{c|c|c|c|c} -5 & -4 & -1 & -4 & -5 \end{array}$$

$$\begin{array}{c|c|c} 2 & 2 & 2 \\ \hline \end{array}$$
$$\begin{array}{c|c|c} -2 & -2 & -2 \end{array}$$

Here for the first time we have two inequivalent forms $x^2 - 5y^2$ and $2x^2 + 2xy - 2y^2$ with neither being the negative of the other.

4. (a) For positive elliptic forms of discriminant $\Delta = -D$, verify that the smallest value of D for which there are at least two inequivalent forms of discriminant $-D$ is $D = 12$.

<u>*Solution*</u>: We want to find all triples (a,h,c) satisfying $4ac = h^2 + D$, with $0 \le h \le a \le c$ and $h^2 \le D/3$. Every such triple (a,h,c) gives a form $ax^2 + hxy + cy^2$ of discriminant $-D$, and our general theory tells us that two different triples (a,h,c) satisfying these conditions always give forms that are inequivalent.

Since Δ is always congruent to 0 or 1 modulo 4, the smallest values of D to try are $D = 3, 4, 7, 8, 11, 12$.

$D = 3$:

h	ac	(a,c)
1	1	$(1,1)$

$D = 4$:

h	ac	(a,c)
0	1	$(1,1)$

$D = 7$:

h	ac	(a,c)
1	2	$(1,2)$

$D = 8$:

h	ac	(a,c)
0	2	$(1,2)$

$D = 11$:

h	ac	(a,c)
1	3	$(1,3)$

$D = 12$:

h	ac	(a,c)
0	3	$(1,3)$
2	4	$(2,2)$

The first time we get more than one form is for $D = 12$, where we have the two inequivalent forms $x^2 + 3y^2$ and $2x^2 + 2xy + 2y^2$.

(b) If we add the requirement that neither of the two inequivalent forms is a constant multiple of some other form with smaller D, then what is the smallest D?

Solution: If we go one step farther than in part (a) we have:

$D = 15$:

h	ac	(a,c)
1	4	$(1,4)$, $(2,2)$

Thus we have the two forms $x^2 + xy + 4y^2$ and $2x^2 + xy + 2y^2$, and neither is a constant multiple of any other form.

5. Determine all the equivalence classes of positive elliptic forms of discriminants -67, -104, and -347.

Solution: For $D = 67$ we have $4ac = h^2 + 67$. The values of h are odd and satisfy $h^2 \le 67/3$ so the only possibilities are $h = 1$ and $h = 3$.

$D = 67$:

h	ac	(a,c)
1	17	$(1,17)$
3	19	$\cancel{(1,19)}$

The $(1,19)$ entry has to be discarded since it doesn't satisfy $h \le a$. So all forms of discriminant -67 are equivalent to $x^2 + xy + 17y^2$.

For $D = 104$ we have $4ac = h^2 + 104$ with h even and $h^2 \le 104/3$ so $h = 0, 2, 4$. The possibilities are:

$D = 104$:

h	ac	(a,c)
0	26	$(1,26)$, $(2,13)$
2	27	$(3,9)$
4	30	$(5,6)$

Thus we have four equivalence classes, the classes of the forms x^2+26y^2, $2x^2+13y^2$, $3x^2+2xy+9y^2$, and $5x^2+4xy+6y^2$.

For $D = 347$ we have $4ac = h^2 + 347$ with h odd and $h^2 \le 347/3$ so the possibilities are $h = 1, 3, 5, 7, 9$.

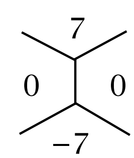
$D = 347$:

h	ac	(a,c)
1	87	$(1,87)$, $(3,29)$
3	89	
5	93	
7	99	$(9,11)$
9	107	

There are just three inequivalent forms, x^2+87y^2, $3x^2+29y^2$, and $9x^2+7xy+11y^2$.

6. (a) Determine all the equivalence classes of 0-hyperbolic forms of discriminant 49.

<u>*Solution*</u>: A 0-hyperbolic form has two regions in its topograph that are labeled 0. If these regions are adjacent to each other, the form is equivalent to a form bxy with $\Delta = b^2$. For $\Delta = 49$ this means $b = 7$ and the form is $7xy$, with a small part of its topograph shown at the right. (Taking $b = -7$ gives an equivalent form.)

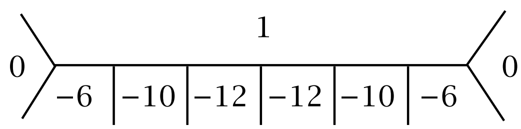


$7xy$

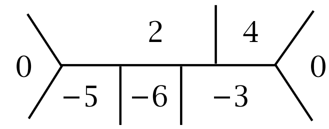
If the two regions with 0 labels are not adjacent, there is at least one separating edge in the topograph, with a positive label on one side and a negative label on the other. These forms can be classified just like in the hyperbolic case. For $\Delta = 49$ we have $h^2 = 49 + 4pq$ so we get the following table:

h	pq	(p,q)
1	12	$(1,12)$, $(2,6)$, $(3,4)$, $(4,3)$, $(6,2)$, $(12,1)$
3	10	$(1,10)$, $(2,5)$, $(5,2)$, $(10,1)$
5	6	$(1,6)$, $(2,3)$, $(3,2)$, $(6,1)$

The first pair $(p,q) = (1,12)$ corresponds to the form $x^2+xy-12y^2$ with topograph shown below. This accounts for all the table entries with $p = 1$.



$x^2 + xy - 12y^2$



$2x^2 + xy - 6y^2$

For $(p,q) = (2,6)$ we have the form $2x^2 + xy - 6y^2$ accounting for the pairs (p,q) with $p = 2$ or 4. The remaining pairs (p,q) are taken care of by the negatives of these two forms, $-x^2 - xy + 12y^2$ and $-2x^2 - xy + 6y^2$. None of these forms are equivalent to each other, as is apparent from their topographs. Thus we have a total of five equivalence classes of forms of discriminant 49, counting the earlier form $7xy$.

(b) Determine which equivalence class in part (a) each of the forms $Q(x,y) = 7xy - py^2$ for $p = 0,1,2,3,4,5,6$ belongs to.

<u>*Solution*</u>: The form $7xy - py^2$ has discriminant 49, for arbitrary p. For $p = 0$ it is the form $7xy$ in part (a). To see which of the forms in part (a) the form $7xy - py^2$ is equivalent to when $p = 1,2,3,4,5,6$, we can compute the values of $7xy - py^2$ at the three pairs $(x,y) = (1,0)$, $(0,1)$, and $(1,1)$. These values are 0, $-p$, and $7 - p$. In the topograph of $7xy - py^2$ these three values will occur in the three regions surrounding a vertex, so all we have to do is look at the topographs shown above (and the negatives of these topographs) to find a vertex where the three values 0, $-p$, and $7 - p$ occur. By inspection the answer is:

$$p = 1 \quad \longrightarrow \quad -x^2 - xy + 12y^2$$

$$p = 2 \quad \longrightarrow \quad -2x^2 - xy + 6y^2$$

$$p = 3 \quad \longrightarrow \quad 2x^2 + xy - 6y^2$$

$$p = 4 \quad \longrightarrow \quad -2x^2 - xy + 6y^2$$

$$p = 5 \quad \longrightarrow \quad 2x^2 + xy - 6y^2$$

$$p = 6 \quad \longrightarrow \quad x^2 + xy - 12y^2$$