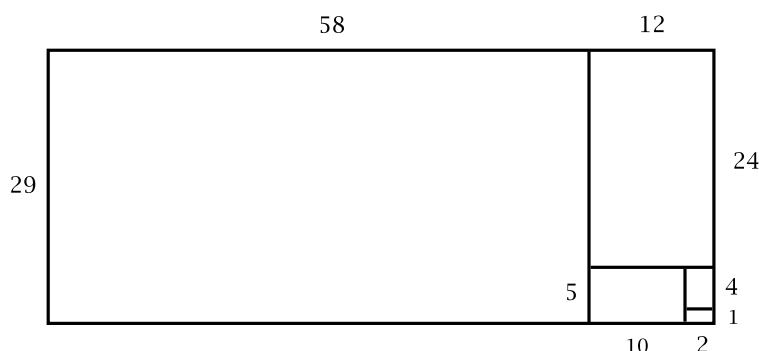


1. In this problem let us call a rectangle whose length equals twice its width a *domino*.
 (a) Show that a 29×70 rectangle can be covered with a finite number of nonoverlapping dominoes of various sizes, with no two dominoes having the same size.

Solution: Here you are supposed to remember how the continued fraction for a rational number p/q gives a decomposition of a $p \times q$ rectangle into squares. We have $29/70 = 1/2 + 1/2 + 1/2 + 1/2 + 1/2$, so the squares occur in pairs, forming dominoes:

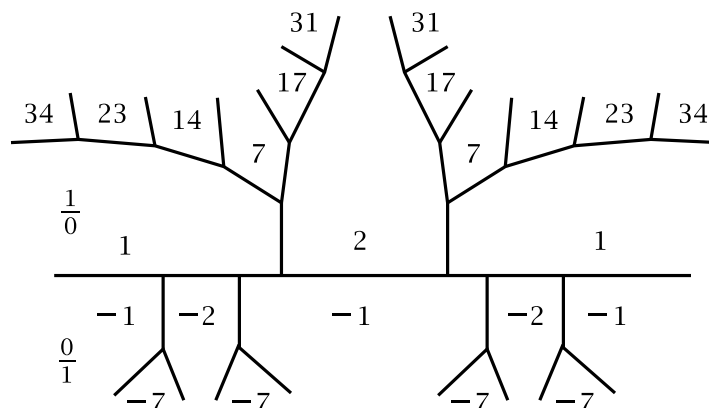


(b) Find an infinite sequence of integer pairs (a, b) with the g.c.d. of a and b being 1, such that an $a \times b$ rectangle can be covered by finitely many nonoverlapping dominoes all of different sizes.

Solution: After part (a) we see that the fractions $a_n/b_n = 1/2 + 1/2 + \cdots + 1/2$ with n terms $1/2$ give rise to $a_n \times b_n$ rectangles that can be covered with n dominoes in the desired way.

2. Find all the numbers less than 40 that occur as differences between a and b in primitive Pythagorean triples (a, b, c) . How many different primitive triples are there for each difference $a - b$ or $b - a$? (Give a reason for your answer.)

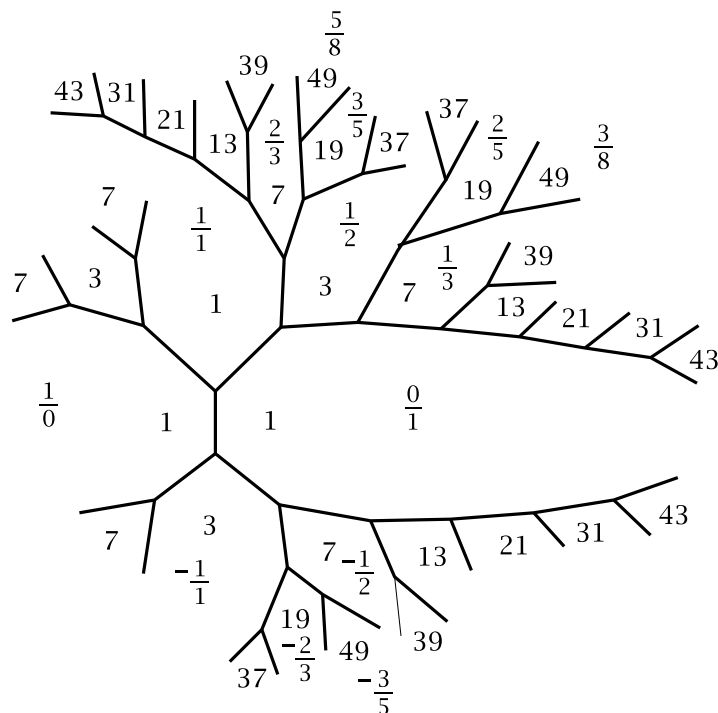
Solution: For primitive Pythagorean triples (a, b, c) we have $a = 2pq$ and $b = p^2 - q^2$ where p and q are of opposite parity and have no common divisors. We want the difference $b - a = p^2 - q^2 - 2pq$ to be a number less than 40 in absolute value, so we look at the topograph of the quadratic form $x^2 - 2xy - y^2$ to find all values between -40 and $+40$. This form has a periodic separator line as shown in the following figure:



All the positive values less than 40 are shown, and the negative values are just the negatives of the positive values. Thus we have the list of values 1, 2, 7, 14, 17, 23, 31, 34 (and their negatives). All these come from pairs (p, q) with no common divisors, but we still need to impose the condition that p and q are not both odd. Notice that $p^2 - q^2 - 2pq$ is even if p and q are both odd, and $p^2 - q^2 - 2pq$ is odd if p and q have opposite parity. Thus we want only the odd values in the topograph: 1, 7, 17, 23, 31. These are the numbers less than 40 that occur as differences between a and b in primitive Pythagorean triples (a, b, c) . Since the topograph is periodic, each of the differences 1, 7, 17, 23, 31 occurs for infinitely many different choices of p and q , with larger and larger values of p and q . Hence each difference 1, 7, 17, 23, 31 occurs for infinitely many different pairs (a, b) since the number $a = 2pq$ takes infinitely many different values for the larger and larger values of p and q .

3. (a) Find all integer solutions of the equation $x^2 - xy + y^2 = 49$.

Solution: First we look in the topograph to find where the value 49 occurs. The topograph has much symmetry so we only draw the portion of the right half of it where values less than 50 occur:



The value 49 occurs three times here, for $(x, y) = \pm(3, 8)$, $\pm(5, 8)$, and $\pm(-3, 5)$. In the other half of the topograph there are the symmetric pairs $(x, y) = \pm(8, 3)$, $\pm(8, 5)$, and $\pm(-5, 3)$. This is 12 solutions so far, and there may be more coming from nonprimitive pairs (x, y) . Namely, the three places where the value 1 occurs in the topograph give addition solutions by multiplying by 7 since $7^2 = 49$. This gives six more solutions $(x, y) = \pm(7, 0)$, $\pm(0, 7)$, and $\pm(7, 7)$.

(b) Find formulas for all rational points on the ellipse $x^2 - xy + y^2 = 1$.

Solution: We can proceed just as we did for rational points on the circle $x^2 + y^2 = 1$. As an initial rational point on the ellipse we choose $(0, 1)$. Lines through this point have equations $y = mx + 1$. Substituting this into the equation $x^2 - xy + y^2 = 1$ we get a quadratic equation in x which simplifies to $(m^2 - m + 1)x^2 + (2m - 1)x = 0$. One solution of this is $x = 0$, and the other is $x = (1 - 2m)/(m^2 - m + 1)$. Plugging this into $y = mx + 1$ and simplifying yields $y = (1 - m^2)/(m^2 - m + 1)$. Thus we have

$$(x, y) = \left(\frac{1 - 2m}{m^2 - m + 1}, \frac{1 - m^2}{m^2 - m + 1} \right)$$

This is a rational point whenever m is rational, and conversely, if x and y are rational then so is m , from the equation $y = mx + 1$. Thus the rational points on the ellipse are given by the preceding displayed formula as m ranges over all rational values. The value $m = 1/2$ gives the initial point $(0, 1)$ and the point $(0, -1)$ is the limiting value of the formula as m approaches ∞ .

We could also write $m = p/q$ and then the formula become

$$(x, y) = \left(\frac{q^2 - 2pq}{p^2 - pq + q^2}, \frac{q^2 - p^2}{p^2 - pq + q^2} \right)$$

(c) Find formulas for all integer solutions of the equation $a^2 - ab + b^2 = c^2$.

Solution: As we did for Pythagorean triples, we divide the equation $a^2 - ab + b^2 = c^2$ by c^2 to get $(\frac{a}{c})^2 - (\frac{a}{c})(\frac{b}{c}) + (\frac{b}{c})^2 = 1$. Thus we want rational points on the ellipse $x^2 - xy + y^2 = 1$, where $x = \frac{a}{c}$ and $y = \frac{b}{c}$. The formulas in (b) give solutions

$$(a, b, c) = (q^2 - 2pq, q^2 - p^2, p^2 - pq + q^2)$$

These give all the solutions up to multiplication of each of the formulas for a , b , and c by a constant k . This is the issue of reducing the fractions $\frac{a}{c}$ and $\frac{b}{c}$ to lowest terms. There are subtleties here which I did not intend for you to get into, and the grading of this problem will ignore these subtleties. For example, the solution $(a, b, c) = (-3, -3, 3)$ is obtained from the formulas by taking $(p, q) = (2, 1)$ but the solution $(-1, -1, 1)$ cannot be obtained from the formulas for any integer values of p and q , but only by multiplying the $(-3, -3, 3)$ solution by the constant $k = 1/3$.

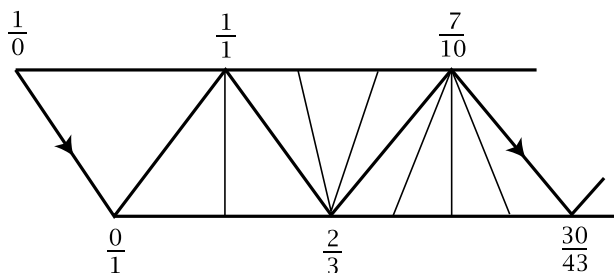
4. Find all integer solutions of $85x + 271y = 1$.

Solution: First we compute the continued fraction for $85/271$. We have $271/85 = 3 + 16/85$, $85/16 = 5 + 5/16$, $16/5 = 3 + 1/5$, so the continued fraction is $85/271 = 1/\frac{1}{3} + 1/\frac{1}{5} + 1/\frac{1}{3} + 1/\frac{1}{5}$. The next-to-last convergent is $1/\frac{1}{3} + 1/\frac{1}{5} + 1/\frac{1}{3} = 16/51$, so there is an edge in the Farey diagram from $16/51$ to $85/271$. Thus the matrix $\begin{pmatrix} 16 & 85 \\ 51 & 271 \end{pmatrix}$ has determinant ± 1 , and in fact the determinant is $+1$, so $16 \cdot 271 - 51 \cdot 85 = 1$. Hence one solution of $85x + 271y = 1$ is $(x, y) = (-51, 16)$. The general solution is then $(x, y) = (-51 + 271k, 16 - 85k)$ for k an arbitrary integer.

5. Using the Farey diagram, find the value of the periodic continued fraction

$$\overline{1/\frac{1}{1} + 1/\frac{1}{2} + 1/\frac{1}{3} + 1/\frac{1}{4}}$$

Solution: The infinite periodic strip of triangles in the Farey diagram for this periodic continued fraction has the form



with this block repeated infinitely often. The periodicity is given by the translation which is the linear fractional transformation taking the edge $\langle 1/0, 0/1 \rangle$ to the edge $\langle 7/10, 30/43 \rangle$. The matrix of this transformation is $\begin{pmatrix} 7 & 30 \\ 10 & 43 \end{pmatrix}$ so the transformation is $T(z) = (7z + 30)/(10z + 43)$. Note that the matrix of T has determinant $+1$ so it preserves orientation and hence is the translation we want. The value of the continued fraction will be the positive root of the equation $(7z + 30)/(10z + 43) = z$. This simplifies to $7z + 30 = 10z^2 + 43z$ or $10z^2 + 36z - 30 = 0$ or $5z^2 + 18z - 15 = 0$, with positive root $z = (-9 + 2\sqrt{39})/5$, so this is the value of the continued fraction.

6. (a) Compute one period of the periodic separator line in the topograph for the form $Q(x, y) = x^2 - 29y^2$.

Solution: Here is half of a period:

$1/0$						$11/2$	$27/5$						
1						7	5	4	13	20	25	28	29
-29	-28	-25	-20	-13	-4	-5	-7	-1					
$0/1$						$5/1$	$16/3$		$70/13$				

The other half is the mirror image of this, going from the value 29 on the right back down to 1.

(b) Find the continued fraction for $\sqrt{29}$.

Solution: There is a glide reflection symmetry along the separator line taking the $(1, -29)$ pair on the left to the $(29, -1)$ pair on the right, so the continued fraction for $\sqrt{29}$ is $5 + \overline{\frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{10}}$. We can also read this off from the RL sequence which is $R^5 \overline{L^2 R L R^2 L^{10} R^2 L R L^2 R^{10}}$.

(c) Find the smallest positive integer solutions for the equations $x^2 - 29y^2 = 1$ and $x^2 - 29y^2 = -1$, if these equations do in fact have such solutions.

Solution: Along the separator line the value -1 first appears at $x/y = 70/13$ so the smallest positive solution of $x^2 - 29y^2 = -1$ is $(x, y) = (70, 13)$. To obtain the smallest positive solution of $x^2 - 29y^2 = +1$ we apply the glide reflection to $(70, 13)$.

The glide reflection matrix is $\begin{pmatrix} p & dq \\ q & p \end{pmatrix} = \begin{pmatrix} 70 & 29 \cdot 13 \\ 13 & 70 \end{pmatrix} = \begin{pmatrix} 70 & 377 \\ 13 & 70 \end{pmatrix}$ and we have $\begin{pmatrix} 70 & 377 \\ 13 & 70 \end{pmatrix} \begin{pmatrix} 70 \\ 13 \end{pmatrix} = \begin{pmatrix} 9801 \\ 1820 \end{pmatrix}$. So the smallest positive solution of $x^2 - 29y^2 = 1$ is $(x, y) = (9801, 1820)$.

7. (a) Use a quadratic form to find the continued fraction expansions of $(9 + \sqrt{3})/26$ and $(9 - \sqrt{3})/26$.

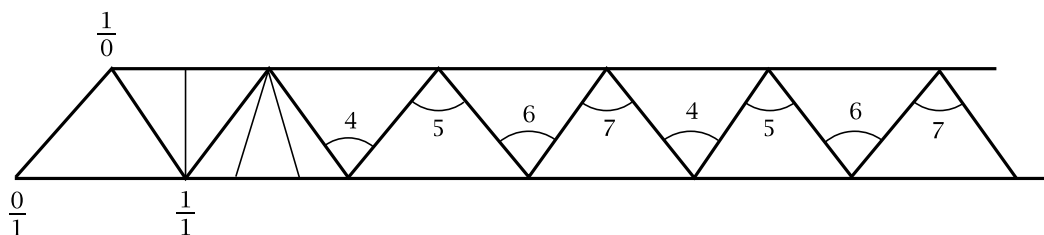
Solution: The numbers $(9 \pm \sqrt{3})/26$ are roots of $26x^2 - 18x + 3 = 0$. We compute enough of the topograph of $26x^2 - 18xy + 3y^2$ to find the periodic separator line:

$$\begin{array}{cccccccc|cccccc}
 & -1 & & -1 & & -1 & & -1 & & -1 & & -1 & \\
 2 & 3 & 2 & 3 & 2 & 3 & 2 & & 2 & 3 & 2 & 3 & 2 \\
 & & & & & & & 11 & & & & & \\
 & & & & & & & 26 & 3 & 0/1 & & &
 \end{array}$$

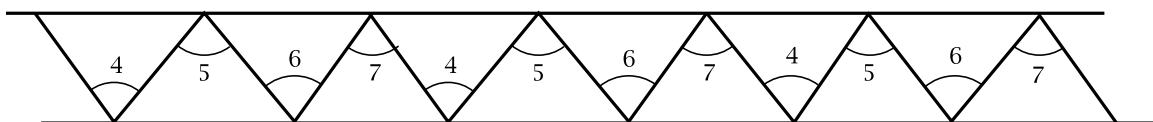
Both roots are positive, and the smaller root corresponds to going out the periodic separator line toward the right since we are in the upper half of the Farey diagram where smaller numbers are to the right. The pattern of left-right sideroads for the path that starts at the edge between $1/0$ and $0/1$ and goes out to the right on the separator line is $LLL\overline{LRR}$ so the continued fraction is $(9 - \sqrt{3})/26 = \frac{1}{3} + \frac{1}{1} + \overline{\frac{1}{1} + \frac{1}{2}}$. The larger root $(9 + \sqrt{3})/26$ corresponds to going out the separator line toward the left. The pattern of left-right sideroads for the path that starts in the same place but goes out to the left on the separator line is $LLR\overline{LLR}$ so the continued fraction is $(9 + \sqrt{3})/26 = \frac{1}{2} + \frac{1}{2} + \overline{\frac{1}{2} + \frac{1}{1}}$.

(b) Terminology: two quadratic irrational numbers α and β are called *conjugates* of each other if they are both roots of the same quadratic equation with rational (or integer) coefficients. For the quadratic irrational $\alpha = 1 + \frac{1}{2} + \frac{1}{3} + \overline{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}$ find the continued fraction for its conjugate β without actually computing the value of α (which would be rather complicated to do).

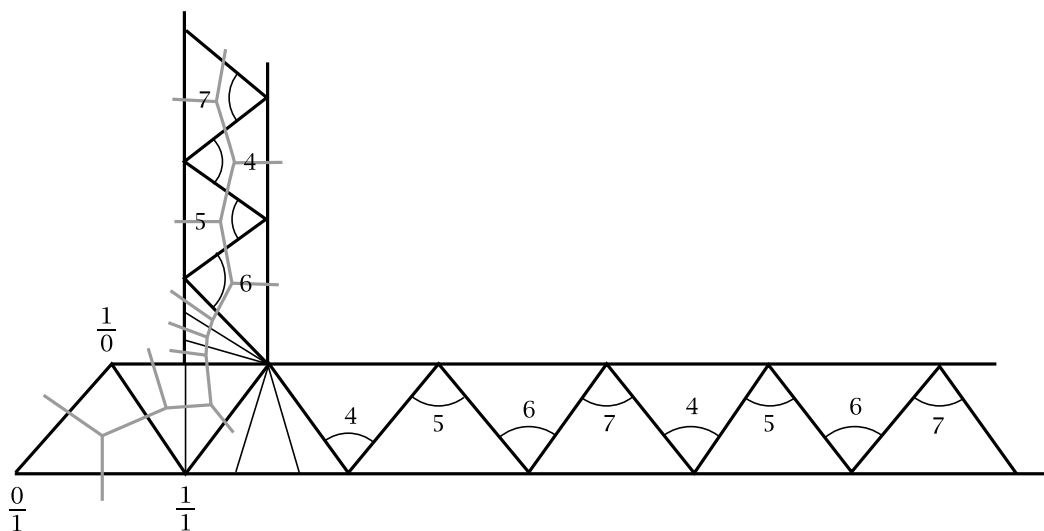
Solution: The strip of triangles for $1 + \frac{1}{2} + \frac{1}{3} + \overline{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}$ is:



We would like to extend the periodic part of this strip so that it is periodic in both directions:



The following diagram shows how this is done:



Thus the fan of 3 triangles and half the fan of two triangles in the original strip form part of a fan of 7 triangles in the periodic strip. From this we can read off that the sequence of “side roads” for the conjugate number β is $RLRL^3\overline{R^6L^5R^4L^7}$. Thus we have

$$\beta = 1 + \nearrow_1 + \nearrow_1 + \nearrow_3 + \overline{\nearrow_6 + \nearrow_5 + \nearrow_4 + \nearrow_7}$$

8. Find the linear fractional transformation that interchanges the two ends of the edge $\langle a/b, c/d \rangle$ of the Farey diagram and preserves orientation of the diagram.

Solution: We want a linear fractional transformation that takes the edge $\langle a/b, c/d \rangle$ to $\langle c/d, a/b \rangle$ and has determinant $+1$. Our usual way of doing this would be to note that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ takes $\langle 1/0, 0/1 \rangle$ to $\langle a/b, c/d \rangle$ and $\begin{pmatrix} c & a \\ d & b \end{pmatrix}$ takes $\langle 1/0, 0/1 \rangle$ to $\langle c/d, a/b \rangle$ so the product $\begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1}$ takes $\langle a/b, c/d \rangle$ to $\langle c/d, a/b \rangle$. However this product has determinant -1 instead of $+1$ since the two matrices $\begin{pmatrix} c & a \\ d & b \end{pmatrix}$ and $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ differ by switching two columns, which changes the sign of the determinant, and taking the inverse matrix doesn’t change the sign of the determinant. To correct this problem all we need to do is change the sign of one column of the original two matrices. This leads to a final answer of

$$\begin{pmatrix} -c & a \\ -d & b \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \begin{pmatrix} -c & a \\ -d & b \end{pmatrix} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} -ab - cd & a^2 + c^2 \\ -b^2 - d^2 & ab + cd \end{pmatrix}$$

Changing the sign of a different column of the original two matrices could lead to a final matrix which is the negative of this one, and this would also be a correct answer since a matrix and its negative give the same linear fractional transformation.