Rules: You can use your notes for the course and any of the books listed on the course webpage, but no other written sources are allowed (including things on the web). You can talk to the instructor about the problems (to clarify the meaning of the problems, if that is necessary), but no one else. Your solutions to the problems should not quote results about smooth manifolds that weren't covered in the class. If in doubt about whether a particular result can be quoted, check with the instructor (email is OK).

1. Let M be a smooth manifold (not necessarily connected) and let $f:U \to V$ be a diffeomorphism between two open sets in M whose closures are disjoint. Let N be the quotient space of M under the identifications $x \sim f(x)$ for all $x \in U$. Show that if N is Hausdorff then it is a smooth manifold such that the quotient map $q:M \to N$ is a local diffeomorphism (i.e., a diffeomorphism when restricted to a suitably small neighborhood of any point in M.)

2. (a) Let M be a smooth connected manifold with N a smooth submanifold such that $\dim N \leq \dim M - 2$. Show that $M - N$ is connected.

(b) Show that if M is a connected smooth manifold, then any two points in M lie in a connected one-dimensional smooth embedded submanifold of M.

3. Let $f:M \to N$ be a smooth map between smooth manifolds M and N and let $x \in N$ be a regular value of f. We know then that the preimage $f^{-1}(x)$ is a smooth submanifold $P \subset M$. Show that the normal bundle of P in M is trivial (i.e., isomorphic to the product vector bundle).

4. This problem is about a mathematical model for mechanical linkages in the plane. Such a linkage consists of a finite number of points $z_1, \cdots, z_p \in \mathbb{R}^2$ called "pivot points" together with a finite number r of rods joining certain pairs (z_i, z_j), such a rod having a specified length L_{ij}. We also allow certain of the pivot points to be pinned down to fixed points in the plane, so only the other pivot points are allowed to move. With all this data specified, the set of all such linkages forms a certain subspace M of $\mathbb{R}^{2k}$ where k is the number of unpinned pivot points.

(a) Show that M is a smooth manifold except for a set of r-tuples of lengths L_{ij} of measure 0 in $\mathbb{R}^r$, and compute the dimension of this manifold M.

(b) Show that the manifolds M that arise in part (a) from the nonexceptional r-tuples of lengths are always orientable. (See problem 3.)

(c) As a special case, consider linear linkages, where each pivot point z_i is joined to the next one z_{i+1} by a rod, and there are no other rods.

(i) What is M when only the first pivot point z_1 is pinned down?

(ii) When only the first and last pivot points z_1 and z_p are pinned down, show that M is a sphere S^{p-3} when the distance from z_1 to z_p is just slightly less than the total length of all the rods. (Hint: use induction on p and the fact that S^n is the union of all the spheres S^{n-1} parallel to the equator, plus the limiting cases of the north and south poles.)

(iii) (bonus problem) When $p = 5$ and only the first and last pivot points p_1 and p_5 are pinned down, show that by choosing the lengths of the rods appropriately one can realize each closed orientable surface of genus four or less as the manifold M.

5. As we know, a Lie group is by definition a smooth manifold G together with a group structure on G such that both the group multiplication $G \times G \to G$, $(g,h) \mapsto gh$, and the inversion map $G \to G$, $g \mapsto g^{-1}$, are smooth maps. Show that smoothness of the inversion map follows automatically from the other conditions in the definition. [Hint: Show that the map $\varphi : G \times G \to G \times G$, $\varphi(g,h) = (g, gh)$, is a bijection and a local diffeomorphism, hence is a diffeomorphism. Along the way you may wish to show that the maps $L_g : G \to G$, $L_g(h) = gh$, are diffeomorphisms for each $g \in G$.]

6. The map $f : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $f(x,y,z) = (x^2 - y^2, xy, xz, yz)$ satisfies $f(x,y,z) = f(-x,-y,-z)$, so the restriction of f to the unit sphere S^2 induces a map $\overline{f} : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4$. Show this map $\overline{f}$ is a smooth embedding.

7. Let G be a Lie group and let $H \subset G$ be a subgroup which is a smooth submanifold.

(a) Show that H is itself a Lie group.

(b) Show that if H has the same dimension as G, then H is a union of components of G. (If you use the result in a homework problem here, include the proof of that result.)

8. (This is closely related to problem 3.) Show that a smooth map $p : M \to N$ with no critical points is a smooth fiber bundle if M is compact and N is connected.