1. Show that a manifold, according to our definition, has at most countably many components (which are the same as path-components since manifolds are locally path-connected).

2. (a) Show that every homeomorphism $f\colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $D^n \to D^n$. (A simple formula suffices. This construction is called "the Alexander trick".)
(b) Show that if one forms a quotient space X of the disjoint union of two n-balls D_1^n and D_2^n by identifying each $x \in \partial D_1^n$ with its image in ∂D_2^n under an arbitrary homeomorphism $f\colon \partial D_1^n \to \partial D_2^n$, then X is homeomorphic to S^n. (Here the notation ∂D^n means the boundary sphere S^{n-1} of D^n.)
(c) The (topological) Schoenflies Theorem, proved by Morton Brown around 1960, states that if one has a subspace of $\mathbb{R}^n$ that is homeomorphic to S^{n-1} via some topological embedding $f\colon S^{n-1} \to \mathbb{R}^n$ that can be extended to an embedding $F\colon S^{n-1} \times (-1,1) \to \mathbb{R}^n$ with $F(x,0) = f(x)$ for all $x \in S^{n-1}$, then $f(S^{n-1})$ bounds a ball in $\mathbb{R}^n$, i.e., there is an embedding $g\colon D^n \to \mathbb{R}^n$ with $g(S^{n-1}) = f(S^{n-1})$. Assuming this, show that a compact manifold of dimension n that is covered by two charts is homeomorphic to S^n. Here we mean "chart" in the sense of a homeomorphism from an open set in the manifold onto all of $\mathbb{R}^n$. You might also need to use the n-dimensional version of the Jordan curve theorem, which says that the complement of a subspace of $\mathbb{R}^n$ homeomorphic to S^{n-1} has exactly two components.

3. Let's assume you know the fact that every compact connected surface can be obtained as the quotient space of a polygon with an even number of sides by identifying the sides in pairs. (Here the word "polygon" means a compact convex subspace of $\mathbb{R}^2$ bounded by a simple closed curve formed by a finite number of straight line segments. There is no harm in taking just a standard regular polygon with an even number of sides.)
(a) Show conversely that the quotient space of an even-sided polygon obtained by identifying its edges in pairs using an arbitrary pairing of the edges is always a compact connected topological surface.
(b) Show that every compact connected surface can be covered by three coordinate charts, where by "chart" we mean an open set homeomorphic to $\mathbb{R}^2$. (Hint: start by representing the surface as a polygon with edges identified in pairs.)

4. Suppose we cover S^n by the two coordinate charts given by stereographic projection from the points $(0, \cdots, 0, \pm 1)$. Show that this gives S^n a C^∞ structure by finding explicit formulas for the coordinate charts.

5. Suppose we are given real numbers $a < b < c < d$ together with C^∞ functions $f\colon (-\infty, b) \to \mathbb{R}$ and $g\colon (c, \infty) \to \mathbb{R}$. Show that there exists a C^∞ function $h\colon \mathbb{R} \to \mathbb{R}$ which equals f on $(-\infty, a)$ and g on (d, ∞).

6. (a) Show that the map $F\colon \mathrm{int}(D^n) \to \mathbb{R}^n$ defined by $F(x) = x/\sqrt{1-|x|^2}$ is a diffeomorphism with inverse $G(x) = x/\sqrt{1+|x|^2}$.
(b) In the case $n = 2$ draw a picture of the images of the horizontal and vertical lines $x = \mathit{constant}$ and $y = \mathit{constant}$ under the map G. The picture should show how the images of these lines intersect and where their ends are. (You should be able to do this by hand, without the assistance of a computer.)

7. (a) Show that if M is a smooth submanifold of N and N is a smooth submanifold of P, then M is a smooth submanifold of P.
(b) Show that if M is a smooth submanifold of N and P is a smooth submanifold of Q, then $M \times P$ is a smooth submanifold of $N \times Q$.