1. Give details in the proof of the fact, stated in class, that for a composition gf of smooth maps $f: M \to N$ and $g: N \to P$, the induced maps on tangent bundles $Df: TM \to TN$ and $Dg: TN \to TP$ satisfy $D(gf) = Dg \circ Df$. (Here we are using the definition of tangent bundles in terms of coordinate charts.)

2. Show that for smooth manifolds M and N, $T(M \times N)$ is diffeomorphic to $TM \times TN$.

3. (a) Show that if $p_1: E_1 \to M$ and $p_2: E_2 \to M$ are smooth vector bundles, then so is their direct sum $E_1 \oplus E_2 \to M$.

(b) Verify that in a smooth vector bundle $p: E \to M$ the operations of vector addition $E \oplus E \to E$, $(v, w) \mapsto v + w$, and scalar multiplication $\mathbb{R} \times E \to E$, $(t, v) \mapsto tv$, are smooth maps.

(c) Verify that a smooth section of a smooth vector bundle is a smooth embedding.

4. (a) Show that the Klein bottle $K = (S^1 \times [0,1])/(z,0) \sim (\bar{z}, 1)$ has a nonvanishing (i.e., nowhere zero) vector field.

(b) Express the tangent bundle TK as the direct sum $E_1 \oplus E_2$ of two 1-dimensional vector bundles, where E_2 is the trivial bundle. Describe E_1 explicitly.

5. If M is a submanifold of N, show that TM is the subspace of TN consisting of vectors tangent to smooth curves in M.

6. (a) Recall that $\mathbb{R}\mathrm{P}^n$ is the space of lines L in $\mathbb{R}^{n+1}$ passing through the origin. The *canonical line bundle* over $\mathbb{R}\mathrm{P}^n$ is the space $E = \{(L, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \mid v \in L\}$, or loosely speaking, "the vectors in lines through the origin". Show that the projection $p: E \to \mathbb{R}\mathrm{P}^n$, $p(L, v) = L$, defines a vector bundle, i.e., verify the local triviality condition. Hint: think about projecting one line orthogonally onto another.

(b) Determine whether the vector bundle E in part (a) is the trivial bundle in the special case $n = 1$. (Give reasons for your answer, of course.)

(c) Define the orthogonal complement $E^\perp = \{(L, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \mid v \perp L\}$ with projection $p: E^\perp \to \mathbb{R}\mathrm{P}^n$ sending (L, v) to L. Show that this too is a vector bundle, and determine whether it is trivial when $n = 1$.

(d) Show that $E \oplus E^\perp$ is isomorphic to the trivial bundle $\mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$.

7. (a) Regarding S^n as the unit sphere in $\mathbb{R}^{n+1}$ as usual, let E be the quotient space of $TS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid v \perp x\}$ under the identifications $(x,v) \sim (-x,-v)$. Give an explanation for why E is the tangent bundle $T\mathbb{R}\mathrm{P}^n$. (This implies in particular that E is a vector bundle over $\mathbb{R}\mathrm{P}^n$.)

(b) Show that $T\mathbb{R}\mathrm{P}^n$ is the trivial bundle for $n = 1, 3$, following the same line of reasoning as for TS^n in these dimensions. (This holds also for $n = 7$ using octonions, but let's skip that case.)

(c) One might guess that $T\mathbb{R}\mathrm{P}^n$ is the same as the bundle $E^\perp$ in the previous problem. Show that this is false for $n = 1$. (Later we will be able to show it is false for all odd n.)

(d) Let E' be the quotient space of the normal bundle

$$NS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid v = tx \ \text{ for some } t \in \mathbb{R}\}$$

under the identifications $(x,v) \sim (-x,-v)$. Show that E' is the trivial bundle $\mathbb{R}\mathrm{P}^n \times \mathbb{R}$.

Bonus problem — it's a little tricky:

(e*) Show that $T\mathbb{R}\mathrm{P}^n \oplus E'$ is the direct sum of $n+1$ copies of the canonical line bundle over $\mathbb{R}\mathrm{P}^n$ defined in the previous problem. Hint: consider the quotient of $S^n \times \mathbb{R}^{n+1}$ under the identifications $(x,v) \sim (-x,-v)$. Show this is the direct sum of $n+1$ copies of the bundle $(S^n \times \mathbb{R})/(x,v) \sim (-x,-v)$, and show this 1-dimensional bundle over $\mathbb{R}\mathrm{P}^n$ is isomorphic to the canonical line bundle by thinking of $S^n \times \mathbb{R}$ as the normal bundle NS^n.