1. Show that if two manifolds M and N are diffeomorphic then their unit tangent bundles T_1M and T_1N are also diffeomorphic. (Assume that M and N are embedded in Euclidean spaces, so it makes sense to talk about length of tangent vectors and we can define T_1M to be the subspace of TM consisting of tangent vectors of length 1, and likewise for T_1N. We showed in class that these are smooth manifolds.)

2. Given a vector bundle $p\colon E \to B$ we can define its projectivization $P(E)$ to be the space of lines through the origin in fibers of E, topologized as a quotient space of the complement of the zero section of E, identifying nonzero vectors which are scalar multiples of each other. Show that for a smooth manifold M the projectivization $PT(M)$ of its tangent bundle is again a smooth manifold.

3. Show that for an arbitrary n-manifold M, orientable or not, the tangent bundle TM is always orientable as a manifold of dimension $2n$. (This is different from saying TM is orientable as a vector bundle.)

4. Give an example of two nonorientable vector bundles $E_1 \to B$ and $E_2 \to B$, with B path-connected, such that $E_1 \oplus E_2$ is orientable, and also give an example where $E_1 \oplus E_2$ is nonorientable. (Both cases can be done with E_1 and E_2 being 1-dimensional bundles.)

5. (a) Let $GL_n(\mathbb{R})$ be the space of invertible $n \times n$ matrices with entries in $\mathbb{R}$, topologized as a subspace of $\mathbb{R}^{n^2}$, thinking of the entries of $n \times n$ matrices as coordinates in $\mathbb{R}^{n^2}$. Show that $GL_n(\mathbb{R})$ has exactly two path-components, consisting of matrices with determinant > 0 and < 0, respectively. Hint: Invertible matrices can be diagonalized by elementary row operations consisting of adding a scalar multiple of one row to another. Show that each of these operations can be realized by a path in $GL_n(\mathbb{R})$.

(b) Using part (a), show that every orientable vector bundle $E \to S^1$ is trivial. Do this also when S^1 is replaced by any finite graph. (Finiteness isn't actually needed, but assume it for simplicity.)

6. If M is an open Möbius band (i.e., M does not include its boundary circle), show that M is nonorientable by showing that TM is nonorientable. (It suffices to show the restriction of TM to the core circle of M is nonorientable.)

7. Let $E \to M$ be a vector bundle over an n-manifold M with $n > 1$. Show that E is orientable if its restriction over the complement of a point in M is orientable. Taking $E = TM$, this shows that M is orientable if (and only if) $M - \{point\}$ is orientable. Why is the assumption $n > 1$ in the first sentence necessary?

8. Recall that the connected sum $M_1 \# M_2$ of two n-manifolds M_1 and M_2 is defined by deleting a point from each of M_1 and M_2 and then gluing neighborhoods of these deleted points (which are diffeomorphic to $S^{n-1} \times \mathbb{R}$) together in a certain way. Show that $M_1 \# M_2$ is orientable if and only if both M_1 and M_2 are orientable. (You can assume $n > 1$ since all 1-manifolds are orientable.)