1. Show that if M is a smooth submanifold of a manifold Q then a map $f: M \to N$ is smooth if and only if for each point $x \in M$ there is an open neighborhood U of x in Q and a smooth map $U \to N$ that agrees with f on $M \cap U$.

2. Given a connected manifold M and two points $x, y \in M$ show that there exists a diffeomorphism $f: M \to M$ with $f(x) = y$, and with f equal to the identity outside some compact set in M. More generally, given two sets $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ of n distinct points in M, show there is a compactly supported diffeomorphism $f: M \to M$ with $f(x_i) = y_i$ for each i, provided that the dimension of M is at least 2. [Updated to include this dimension condition.]

3. Let $p: E \to M$ be a smooth vector bundle. For a closed set K in M, let $s: K \to E$ be a smooth section, so ps is the identity map on K. Show that s can be extended to a smooth section $M \to E$.

4. For a space X, let $\kappa(X)$ denote the set of path-components of X. Show that if X is a Lie group (or more generally a topological group) then $\kappa(X)$ inherits a group structure from X such that the natural projection $X \to \kappa(X)$ sending $x \in X$ to its path-component $[x] \in \kappa(X)$ is a homomorphism. Deduce from this that the path-component of X containing the identity element of X is a normal subgroup, and give a direct proof of this fact that does not use the group $\kappa(X)$.

5. Suppose that G is a connected Lie group and H is a subgroup that contains an open neighborhood of the identity element of G. Show that $H = G$.

6. (a) Show that a discrete normal subgroup H of a connected Lie group G must lie in the center of G. (Here "discrete" means discrete as a subspace of G, i.e., the subspace topology on H is the discrete topology.)

(b) Show that if G is a Lie group and H is a finite normal subgroup then the quotient group G/H is a Lie group. (Don't forget to check that G/H is Hausdorff.)

(c) Let $PGL_n(\mathbb{R}) = GL_n(\mathbb{R})/H$ where H is the subgroup of matrices which are a scalar multiple of the identity matrix. Show that $PGL_n(\mathbb{R})$ is a Lie group of dimension $n^2 - 1$. [Hint: first show that $GL_n(\mathbb{R})$ can be replaced by the subgroup consisting of matrices of determinant ± 1.]

(d) How many path-components does $PGL_n(\mathbb{R})$ have? Also, how does $PGL_n(\mathbb{R})$ compare with $PSL_n(\mathbb{R})$ where we start with $SL_n(\mathbb{R})$ instead of $GL_n(\mathbb{R})$?

(e) For each integer $k > 1$ find a normal subgroup of $U(n)$ which is cyclic of order k.