CW Approximation

A map $f:X\to Y$ is called a **weak homotopy equivalence** if it induces isomorphisms $\pi_n(X,x_0)\to\pi_n(Y,f(x_0))$ for all $n\ge 0$ and all choices of basepoint x_0. Whitehead's theorem can be restated as saying that a weak homotopy equivalence between CW complexes is a homotopy equivalence. It follows easily that this holds also for spaces homotopy equivalent to CW complexes. In general, however, weak homotopy equivalence is strictly weaker than homotopy equivalence. For example, there exist noncontractible spaces whose homotopy groups are all trivial, such as the 'quasi-circle' according to an exercise at the end of this section, and for such spaces a map to a point is a weak homotopy equivalence that is not a homotopy equivalence.

We will show that for every space X there is a CW complex Z and a weak homotopy equivalence $f:Z\to X$. Such a map $f:Z\to X$ is called a **CW approximation** to X. A weak homotopy equivalence induces isomorphisms on all homology and cohomology groups, as we will see, so CW approximations allow many general statements in algebraic topology to be reduced to the case of CW complexes, where one can often make cell-by-cell arguments.

The construction of a CW approximation $f:Z\to X$ for a space X is inductive, so let us describe the induction step. Suppose given a CW complex A with a map $f:A\to X$ and suppose we have chosen a basepoint 0-cell a_γ in each component of A. Then for an integer $k\ge 0$ we will attach k-cells to A to form a CW complex B with a map $f:B\to X$ extending the given f, such that:

(∗) The induced map $f_*:\pi_i(B,a_\gamma)\to\pi_i(X,f(a_\gamma))$ is injective for $i=k-1$ and surjective for $i=k$, for all a_γ.

There are two steps to the construction:

(1) Choose maps $\varphi_\alpha:(S^{k-1},s_0)\to(A,a_\gamma)$ representing a set of generators for the kernel of $f_*:\pi_{k-1}(A,a_\gamma)\to\pi_{k-1}(X,f(a_\gamma))$ for all the basepoints a_γ. We may assume the maps φ_α are cellular, where S^{k-1} has its standard CW structure with s_0 as 0-cell. Attaching cells e^k_α to A via the maps φ_α then produces a CW complex, and the map f extends over these cells using nullhomotopies of the compositions $f\varphi_\alpha$, which exist by the choice of the φ_α's.

(2) Choose maps $f_\beta:S^k\to X$ representing generators for the groups $\pi_k(X,f(a_\gamma))$, attach cells e^k_β to A via the constant maps at the appropriate basepoints a_γ, and extend f over the resulting spheres S^k_β via the f_β's.

The surjectivity condition in (∗) then holds by construction. For the injectivity condition, an element of the kernel of $f_*:\pi_{k-1}(B,a_\gamma)\to\pi_{k-1}(X,f(a_\gamma))$ can be represented by a cellular map $h:S^{k-1}\to B$. This has image in A, so is in the kernel of $f_*:\pi_{k-1}(A,a_\gamma)\to\pi_{k-1}(X,f(a_\gamma))$ and hence is homotopic to a linear combination of the φ_α's, which are nullhomotopic in B, so h is nullhomotopic as well. When $k=1$ there is no group structure on π_{k-1} so injectivity on π_0 does not follow from having a trivial kernel, and we modify the contruction by choosing the cells e^1_α to join each pair of basepoints a_γ that map by f to the same path-component of X. The map f can then be extended over these 1-cells e^1_α.

Note that if the given map $f:A\to X$ happened to be injective or surjective on π_i for some $i < k-1$ or $i < k$, respectively, then this remains true after attaching the k-cells. This is because attaching k-cells does not affect π_i if $i < k-1$, by cellular approximation, nor does it destroy surjectivity on π_{k-1} or indeed any π_i, obviously.

Now to construct a CW approximation $f:Z\to X$ one can start with A consisting of one point for each path-component of X, with $f:A\to X$ mapping each of these points to the corresponding path-component. Having now a bijection on π_0, attach 1-cells to A to create a surjection on π_1 for each path-component, then 2-cells to improve this to an isomorphism on π_1 and a surjection on π_2, and so on for each successive π_i in turn. After all cells have been attached one has a CW complex Z with a weak homotopy equivalence $f:Z\to X$.

Example 4.14. If X is path-connected, this procedure produces a CW approximation having a single 0-cell. A further feature which can be useful is that all the attaching maps for the cells of Z are basepoint-preserving. Thus every connected CW complex is homotopy equivalent to a CW complex with these additional properties.

Example 4.15. One can also apply this technique to produce a CW approximation to a pair (X,X_0). First construct a CW approximation $f_0:Z_0\to X_0$, then starting with the composition $Z_0\to X_0\hookrightarrow X$, attach cells to Z_0 to create a weak homotopy equivalence $f:Z\to X$ extending f_0. It follows from the five-lemma that the map f, regarded as a map of pairs $(Z,Z_0)\to(X,X_0)$, induces isomorphisms on relative as well as absolute homotopy groups.

Here is another application of the technique, giving a more geometric interpretation to the homotopy-theoretic notion of n-connectedness:

Proposition 4.16. *If (X,A) is an n-connected CW pair, then there exists a CW pair $(Z,A)\simeq(X,A)$ rel A such that all cells of $Z-A$ have dimension greater than n.*

Proof: Starting with the inclusion $A\hookrightarrow X$, attach cells to A of dimension $n+1$ and higher to produce a CW complex Z and a map $f:Z\to X$ that is the identity on A and induces an injection on π_n and isomorphisms on all higher homotopy groups. The induced map on π_n is also surjective since this is true for the composition $A\hookrightarrow Z\xrightarrow{f}X$ by the hypothesis that (X,A) is n-connected. In dimensions below n, f induces isomorphisms on homotopy groups since both inclusions $A\hookrightarrow Z$ and $A\hookrightarrow X$ induce isomorphisms in these dimensions. Thus f is a weak homotopy equivalence, and hence a homotopy equivalence by Whitehead's theorem.

To see that f is a homotopy equivalence rel A, form a quotient space W of the mapping cylinder M_f by collapsing each segment $\{a\}\times I$ to a point, for $a\in A$. Assuming f has been made cellular, W is a CW complex containing X and Z as subcomplexes, and W deformation retracts to X just as M_f does. Also, $\pi_i(W,Z)=0$ for all i since f induces isomorphisms on all homotopy groups, so W deformation retracts onto Z. These two deformation retractions of W onto X and Z are stationary on A, hence give a homotopy equivalence $X \simeq Z$ rel A. □

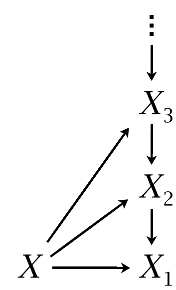
Example 4.17: Postnikov Towers. We can also apply the technique to construct, for each connected CW complex X and each integer $n \geq 1$, a CW complex X_n containing X as a subcomplex such that:

(a) $\pi_i(X_n)=0$ for $i>n$.

(b) The inclusion $X \hookrightarrow X_n$ induces an isomorphism on π_i for $i \leq n$.

To do this, all we have to do is apply the general construction to the constant map of X to a point, starting at the stage of attaching cells of dimension $n+2$. Thus we attach $(n+2)$-cells to X using cellular maps $S^{n+1} \to X$ that generate $\pi_{n+1}(X)$ to form a space with π_{n+1} trivial, then for this space we attach $(n+3)$-cells to make π_{n+2} trivial, and so on. The result is a CW complex X_n with the desired properties.

The inclusion $X \hookrightarrow X_n$ extends to a map $X_{n+1} \to X_n$ since X_{n+1} is obtained from X by attaching cells of dimension $n+3$ and greater, and $\pi_i(X_n)=0$ for $i>n$ so we can apply Lemma 4.7, the extension lemma. Thus we have a commutative diagram as at the right. This is a called a *Postnikov tower* for X. One can regard the spaces X_n as truncations of X which provide successively better approximations to X as n increases. Postnikov towers turn out to be quite powerful tools for proving general theorems, and we will study them further in §4.3.



Now that we have seen several varied applications of the technique of attaching cells to make a map $f:A \to X$ more nearly a weak homotopy equivalence, it might be useful to give a name to the properties that the construction can achieve. To simplify the description, we may assume without loss of generality that the given f is an inclusion $A \hookrightarrow X$ by replacing X by the mapping cylinder of f. Thus, starting with a pair (X,A) where the subspace $A \subset X$ is a nonempty CW complex, we define an ***n*-connected CW model for** (X,A) to be an n-connected CW pair (Z,A) and a map $f:Z \to X$ with $f \mid A$ the identity, such that $f_*:\pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i>n$ and an injection for $i=n$, for all choices of basepoint. Since (Z,A) is n-connected, the map $\pi_i(A) \to \pi_i(Z)$ is an isomorphism for $i<n$ and a surjection for $i=n$. In the critical dimension n, the maps $A \hookrightarrow Z \xrightarrow{f} X$ induce a composition $\pi_n(A) \to \pi_n(Z) \to \pi_n(X)$ factoring the map $\pi_n(A) \to \pi_n(X)$ as a surjection followed by an injection, just as any homomorphism $\varphi:G \to H$ can be factored (uniquely) as a surjection $\varphi:G \to \operatorname{Im}\varphi$ followed by an injection $\operatorname{Im}\varphi \hookrightarrow H$. One can think of Z as a sort of homotopy-theoretic hybrid of A and X. As n increases, the hybrid looks more and more like A, and less and less like X.

Our earlier construction shows:

Proposition 4.13. *For every pair (X,A) with A a nonempty CW complex there exist n-connected CW models $f:(Z,A)\to(X,A)$ for all $n\ge 0$, and these models can be chosen to have the additional property that Z is obtained from A by attaching cells of dimension greater than n.* □

Note that the condition that $Z-A$ consists of cells of dimension greater than n implies that (Z,A) is n-connected, by cellular approximation.

The construction of n-connected CW models involves many arbitrary choices, so it may be somewhat surprising that they turn out to be unique up to homotopy equivalence. This will follow easily from the next proposition. Another application of the proposition will be to build a tower like the Postnikov tower from the various n-connected CW models for a given pair (X,A).

Now continue with Proposition 4.18 in the book.