Something to add to the end of Section 1.2:

Intuitively, loops are one-dimensional and homotopies between them are two-dimensional. Countering this intuition is the phenomenon of space-filling curves. Fortunately such pathology (path-ology?) has little impact in algebraic topology where we are usually free to vary maps by homotopy and eliminate dimension-raising behavior. Here is a concrete result along these lines:

Proposition 1.30. *For a path-connected CW complex X the inclusion $X^1 \hookrightarrow X$ of the* 1*-skeleton induces a surjection on π_1, and the inclusion $X^2 \hookrightarrow X$ induces an isomorphism on π_1.*

For example, since $\pi_1(\mathbb{R}P^2) \approx \mathbb{Z}_2$ and $\mathbb{R}P^2$ is the 2-skeleton of $\mathbb{R}P^n$ in the CW structure described in Example 0.4, we deduce that $\pi_1(\mathbb{R}P^n) \approx \mathbb{Z}_2$ for all $n \geq 2$. The corresponding result for $\mathbb{R}P^\infty$ also holds.

The first statement of the proposition says that if we choose a 0-cell of X as the basepoint, then every loop in X is homotopic to a loop in X^1. In fact, every loop is homotopic to an **edgepath** loop, consisting of a finite sequence of edges, each traversed monotonically. When X has a single 0-cell this follows from the earlier calculation of the fundamental group of a wedge sum of circles. In the general case, since a loop in X^1 has compact image, it lies in a finite subgraph of X^1, and in such a graph every loop is homotopic to an edgepath loop since the fundamental group of a finite connected graph can be computed by choosing a maximal tree as in Example 1.22.

Proof: In view of the preceding proposition we need only prove the second assertion. As a first step let us verify that the inclusion $X^{n-1} \hookrightarrow X^n$ induces an isomorphism on π_1 if $n > 2$. To show this we modify the proof of Proposition 1.26 by replacing the 2-cells e^2_α with n-cells e^n_α. The assumption $n > 2$ makes the spaces A_α simply-connected, so $A \cap B$ is simply-connected. The space B is still contractible, and the isomorphism $\pi_1(X^{n-1}) \approx \pi_1(X^n)$ follows.

By induction this proves the proposition when X is finite-dimensional, so $X = X^n$ for some n. In the infinite-dimensional case, to show that the map $\pi_1(X^2) \to \pi_1(X)$ is surjective, take a loop in X at a basepoint in X^2. This has compact image, so by Proposition A.1 in the Appendix, this image is contained in a finite subcomplex of X and in particular in some X^n. By the finite-dimensional case the map $\pi_1(X^2) \to \pi_1(X^n)$ is surjective, so the loop is homotopic to a loop in X^2. This shows surjectivity of $\pi_1(X^2) \to \pi_1(X)$. For injectivity, a nullhomotopy in X of a loop in X^2 is a map $I \times I \to X$ with compact image lying in some X^n, so since the map $\pi_1(X^2) \to \pi_1(X^n)$ has trivial kernel, the loop must be nullhomotopic in X^2. $\square$