CURVES ON THE 4-PUNCTURED SPHERE: 31 CHARTS

W. Floyd and A. Hatcher

Let $M = M_{g,n}$ be a closed orientable surface of genus g with n points deleted (everything here is in the C^∞ category). We wish to describe some of the global structure of the set $\mathcal{C}(M)$ of isotopy classes of curve systems $C_1 \cup \cdots \cup C_m \subset M$ $(m \geq 1)$ satisfying:

(i) Each C_i is either a circle or a properly embedded arc in M,

(ii) $C_i \cap C_j = \emptyset$ if $i \neq j$, and

(iii) For each i, no component of $M - C_i$ is a 2-cell or an annulus.

Restricting to curve systems containing only circles determines a subset $\mathcal{C}_0(M) \subset \mathcal{C}(M)$.

A further condition we could impose on a curve system $C_1 \cup \cdots \cup C_m$ is

(iv) For each pair $i \neq j$, no component of $M - (C_i \cup C_j)$ is a 2-cell or an annulus (so C_i is not isotopic to C_j).

Curve systems of $\mathcal{C}(M)$ satisfying (iv) form a "basis" for $\mathcal{C}(M)$, in that any element of $\mathcal{C}(M)$ is uniquely representable (up to isotopy) as a formal sum $n_1C_1 + \cdots + n_mC_m$, $n_i \in \mathbb{N}$, where $C_1 \cup \cdots \cup C_m$ satisfies (iv) and "n_iC_i" means "n_i parallel copies of C_i".

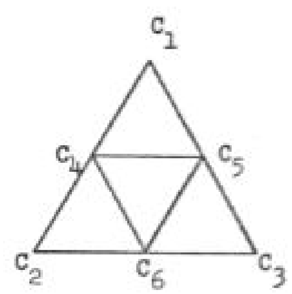
A second algebraic operation is to <u>projectivize</u> by passing to the quotient $\mathcal{C}(M)/\mathbb{N}$, where weighted systems $n_1C_1 + \cdots + n_mC_m$ and $kn_1C_1 + \cdots + kn_mC_m$ are identified for any $k \in \mathbb{N}$ (this is the same as allowing positive rational weights and positive rational scalar multiplication). Instead we will allow positive <u>real</u> weights and positive real scalar multiplication, forming a set $\mathcal{PC}(M)$ containing $\mathcal{C}(M)/\mathbb{N}$ as its "rational" points. $\mathcal{PC}_0(M)$ is defined similarly by using only closed curves.

$\mathcal{PC}(M)$ and $\mathcal{PC}_0(M)$ have the structure of simplicial complexes. The open m-simplices are the projective classes $[n_0C_0 + \cdots + n_mC_m]$ for $[n_0 : \cdots : n_m] \in (\mathbb{R}^+)^{m+1}/\mathbb{R}^+ = \mathrm{int}(\Delta^m)$, with one such m-simplex for each curve system $C_0 + \cdots + C_m$ satisfying (i)-(iv). The $(m-1)$-faces of this m-simplex are obtained by letting one n_i go to zero and then deleting C_i.

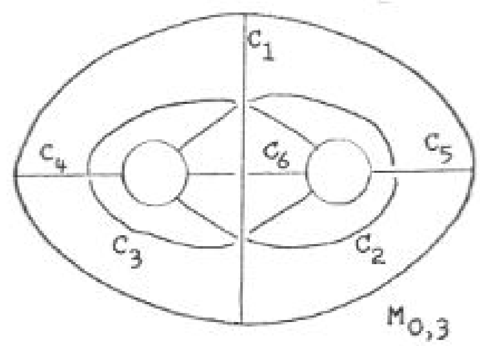
<u>Examples</u>:

$\mathcal{PC}(M_{0,n}) = \emptyset$ for $n \leq 2$, and $\mathcal{PC}(M_{1,0}) = \emptyset$.

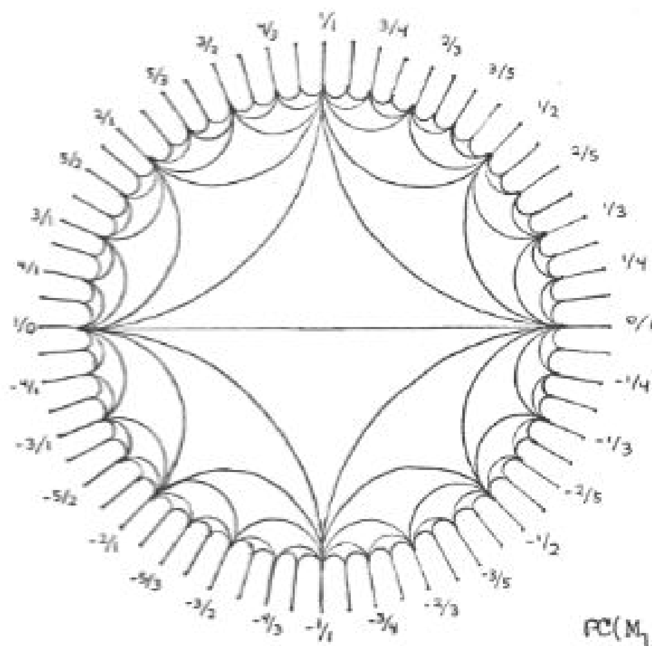
$\mathcal{PC}_0(M_{0,3}) = \emptyset$ and $\mathcal{PC}(M_{0,3})$ is the following 2-complex:



$\mathcal{PC}(M_{0,3})$

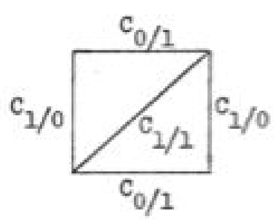


$\mathcal{PC}(M_{1,1})$ is an infinite 2-dimensional complex:

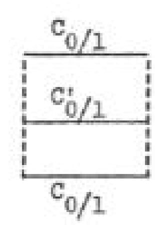


$PC(M_{1,1})$

$PC(M_{1,1})$ is topologically the interior of a 2-disc, with line segments adjoined to all of the "rational" boundary points of the disc. For example, the triangle with vertices at the inner endpoints of the segments labelled 0/1, 1/1, and 1/0 represents the possible weightings of the three disjoint arcs ($C_{0/1}$, $C_{1/1}$, and $C_{1/0}$) of slopes 0/1, 1/1, and 1/0 in $M_{1,1}$.



$M_{1,1}$, a square with vertices deleted and opposite sides identified.



The edge labelled 0/1 represents the weightings of $C_{0/1} \cup C'_{0/1}$, where $C'_{0/1}$ is a circle of slope 0/1 and $C_{0/1}$ is a properly embedded arc of slope 0/1. $PC_0(M_{1,1})$ is $\mathbb{Q} \cup \{1/0\}$, the outer endpoints of the "whiskers".

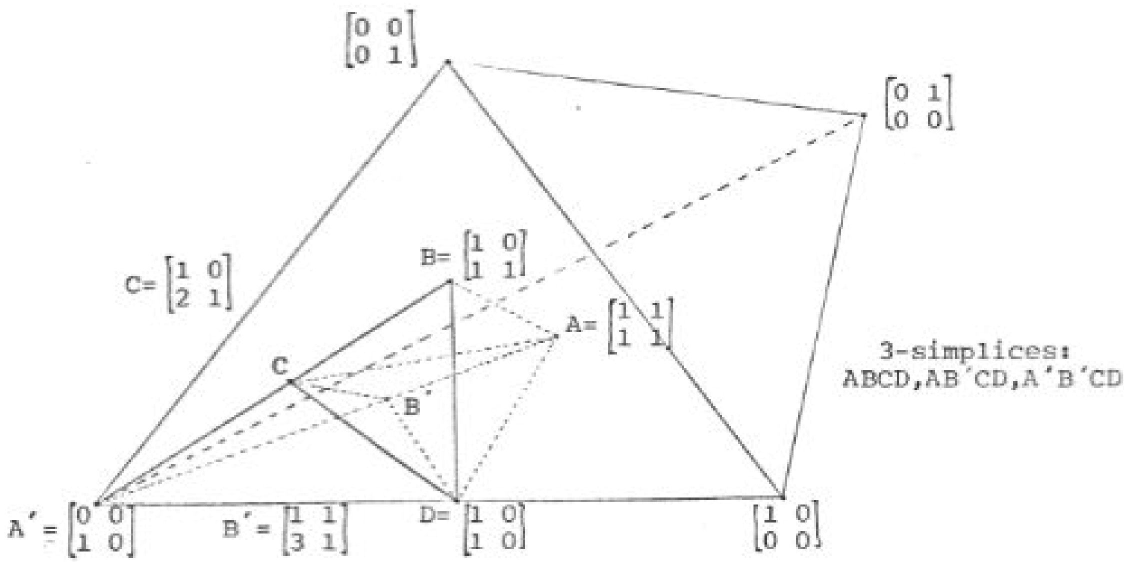
The 4-Punctured Sphere

Besides $M_{1,1}$, the only other surface with $\mathcal{P}C_0(M_{g,n})$ 0-dimensional is $M_{0,4}$. $\mathcal{P}C(M_{0,4})$ is 5-dimensional, but one can get some idea of what it looks like by considering the 2-dimensional slices of $\mathcal{P}C(M_{0,4})$ obtained by fixing the projective class $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \Delta^3$ of the total weights, p, q, r, and s, of the four punctures. [The notation $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ has nothing to do with matrices, but just represents the arrangement of the four punctures of $M_{0,4}$ when $M_{0,4}$ is regarded as $\mathbb{R}^2 \cup \{\infty\}$ minus four points.] Each slice of $\mathcal{P}C(M_{0,4})$ is, like $\mathcal{P}C(M_{1,1})$, topologically the interior of a 2-disc with line segments adjoined to its rational boundary points. However, the cell structures on these slices (the cells being the intersections of the simplices of $\mathcal{P}C(M_{0,4})$ with the 2-dimensional slices) form interesting and intricate patterns, which are shown in the thirty-one charts which follow this commentary. (The line segments attaching to the rational boundary points of the 2-disc are not shown, since they are the same in all slices.) The curve systems representing the simplices are drawn for many of the cells.

Consider, for example, the slice parametrized by $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the barycenter of Δ^3. Elements of $\mathcal{P}C(M_{0,4})$ in this slice have the same total weight at each of the four punctures, and the cell structure on the 2-disc, shown in chart A, is the same as for $\mathcal{P}C(M_{1,1})$. Each 2-cell is a triangle, corresponding to a curve system of six arcs on $M_{0,4}$ joining the four punctures. These six arcs fall into three pairs, the arcs in each pair having the same slope. The three slopes determine the vertices of the given triangle, and deleting one pair of arcs at a time determines the edges of the triangle.

To get a complete set of charts for the rest of the parameter simplex we can restrict to a fundamental domain for the action of the symmetry group S_4 on Δ^3 (this is the permutation group of the four punctures). We choose as a fundamental domain the subtetrahedron of Δ^3 spanned by the points with barycentric (or projective) coordinates $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This subtetrahedron is naturally subdivided into three 3-simplices, with six vertices, twelve edges, and ten faces, for a total of thirty-one simplices. In the interior of each simplex the cell structure of the 2-disc varies only by isotopy, while in passing from the interior of a simplex to a point on its boundary the cell structure changes more drastically: some triangles may shrink to points and some rectangles may flatten to line segments.

Here is a picture of the parameter simplex Δ^3, with its fundamental subtetrahedron subdivided into the three 3-simplices:



Only fifteen of the thirty-one cell-structures on the 2-disc are different: the cell structures corresponding to simplices whose labels differ only by the primes are the same (e.g., the three 3-simplices determine the same cell structure).

All of the cell structures have a great deal of symmetry, since they are invariant under the group G of isotopy classes of diffeomorphisms which take each puncture to itself. If we regard $M_{0,4}$ as $\mathbb{R}^2 - \mathbb{Z}^2/\Gamma$, where Γ is the group generated by 180° rotations of $\mathbb{R}^2$ about the integer lattice points $(\mathbb{Z}^2)$, then G is realized as the subgroup of $PGL(2,\mathbb{Z}) = GL(2,\mathbb{Z})/(\pm I)$ of 2×2 matrices congruent to the identity mod 2 (G has index six in $PGL(2,\mathbb{Z})$). $PGL(2,\mathbb{Z})$ acts on the open 2-disc, viewed as the hyperbolic plane, as hyperbolic isometries (the full isometry group is $PGL(2,\mathbb{R})$), and a fundamental domain for the action of G is exactly one of the triangles in chart A. Hence in any of the other charts the cell structure restricted to such a triangle determines the rest of the cell structure by the action of G, that is, by successive reflections across all of the edges in chart A (the dotted lines in the charts denote the edges of the fundamental triangles).

A Potential Application

Chart A was previously employed in [HT] to help classify the incompressible surfaces in the complement of a 2-bridge knot. In the same vein, charts A, D, and AD were used in [FH] to extend the results of [HT] to 2-bridge links. A quite similar problem, for which all thirty-one of the charts could be used, is to classify the incompressible surfaces in the complement of a 4-strand closed braid B in $S^2 \times S^1$ (B intersecting each $S^2 \times [x]$ in four points). Each transverse intersection of a surface $S \subset S^2 \times S^1 - B$ with a slice $S^2 \times [x]$ determines a point of

$\mathcal{PC}(M_{0,4})$ by ignoring any circles and arcs not satisfying condition (iii). By a preliminary isotopy of S, we may arrange that the number of arcs meeting at each of the four punctures (i.e., the strands of B) be independent of the slice $S^2 \times \{x\}$, so by letting x run around S^1 one would obtain a sequence of points of $\mathcal{PC}(M_{0,4})$ lying in a fixed 2-dimensional slice represented by one of the thirty-one charts. Presumably incompressibility and ∂-incompressibility of the surface S would be equivalent to some sort of minimality condition on the associated edge path in the given chart, as was the case in the situations of [HT] and [FH].

Relationship with Thurston's Projective Lamination Spaces

Laying aside the trivial cases ($M_{0,0}$, $M_{0,1}$, $M_{0,2}$, and $M_{1,0}$) when $\mathcal{PC}(M) = \emptyset$, the surface $M = M_{g,n}$ can be given a metric of constant curvature -1 in which M is complete and has finite area. In this case, a 1-dimensional geodesic submanifold of M which is closed as a subset of M consists of a finite system of curves satisfying (i)-(iv), and each isotopy class of curve systems satisfying (i)-(iv) contains a unique representative consisting of such a geodesic curve system.

Thurston [T] has extended the notion of "geodesic submanifold" to "geodesic lamination": a geodesic lamination L is a closed subset $L \subset M$ which is a disjoint union of (possibly infinitely many) smooth geodesic circles and complete (bi-infinite) geodesic arcs. Thurston next defines the concept of a "transverse measure" for geodesic laminations, generalizing the weighting by positive real numbers of the curves in a curve system satisfying (i)-(iv). He then defines a topology on the resulting set $\mathcal{PL}(M)$ of projective classes of measured geodesic laminations on M, in such a way that the natural injection $\mathcal{PC}(M) \hookrightarrow \mathcal{PL}(M)$ is a topological embedding onto a dense subset of $\mathcal{PL}(M)$. The analog of $\mathcal{PC}_0(M)$ is the subspace $\mathcal{PL}_0(M)$ of projective classes of compactly supported, measured geodesic laminations. $\mathcal{PC}_0(M)$ is a dense subset of $\mathcal{PL}_0(M)$.

$\mathcal{PL}_0(M)$ is homeomorphic to $S^{6g+2n-7}$, and $\mathcal{PL}(M_{g,n})$ is topologically the join of this $S^{6g+2n-7}$ with an $n-1$ dimensional simplex, the projective classes of total weights of the n punctures of $M_{g,n}$. Thus $\mathcal{PL}(M_{g,n})$ is a ball, $B^{6g+3n-7}$, of the same dimension as $\mathcal{PC}(M_{g,n})$ if $n > 0$.

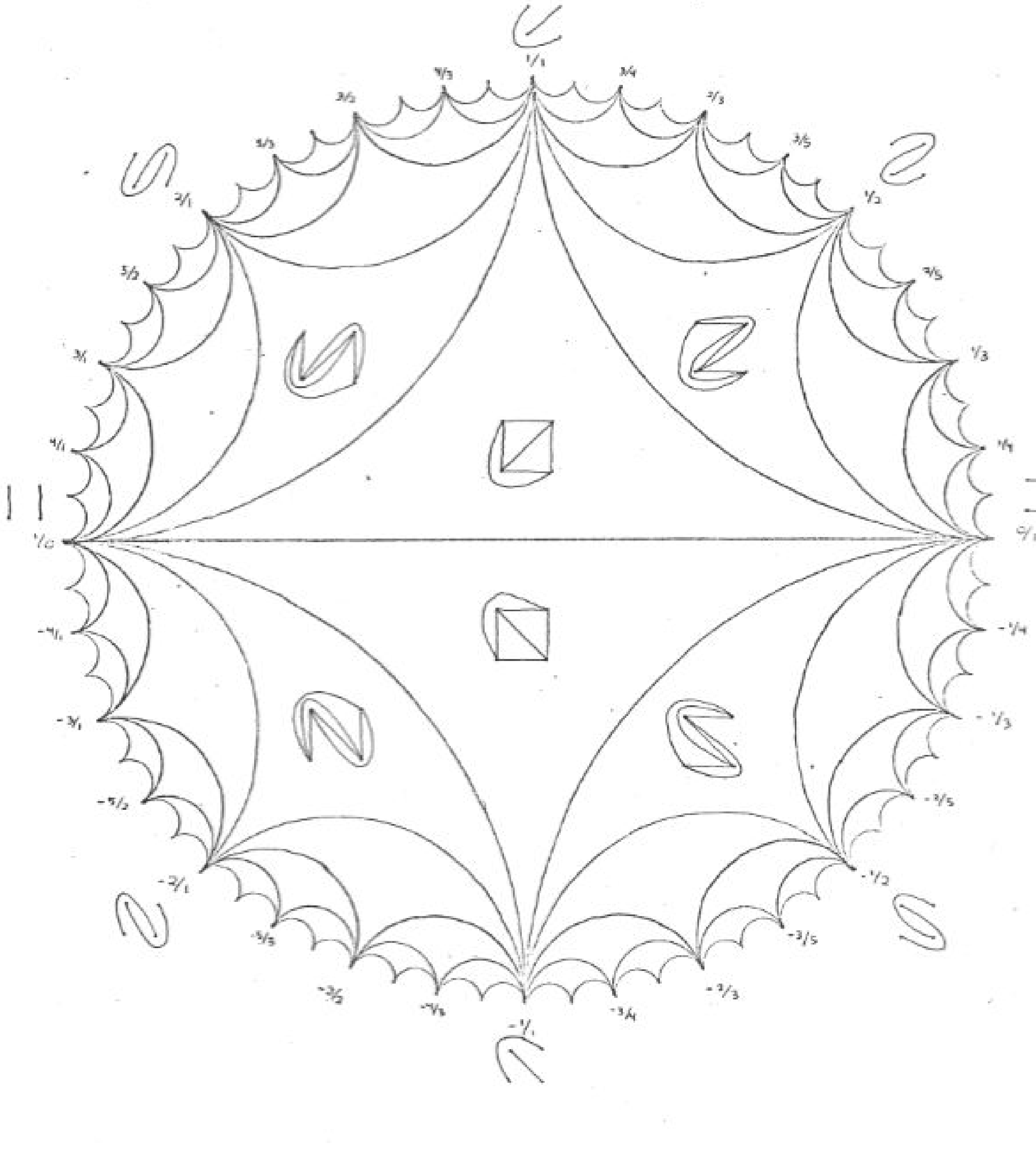
References

[FH] Floyd, W. and Hatcher, A., "Incompressible surfaces in 2-bridge link complements", to appear.

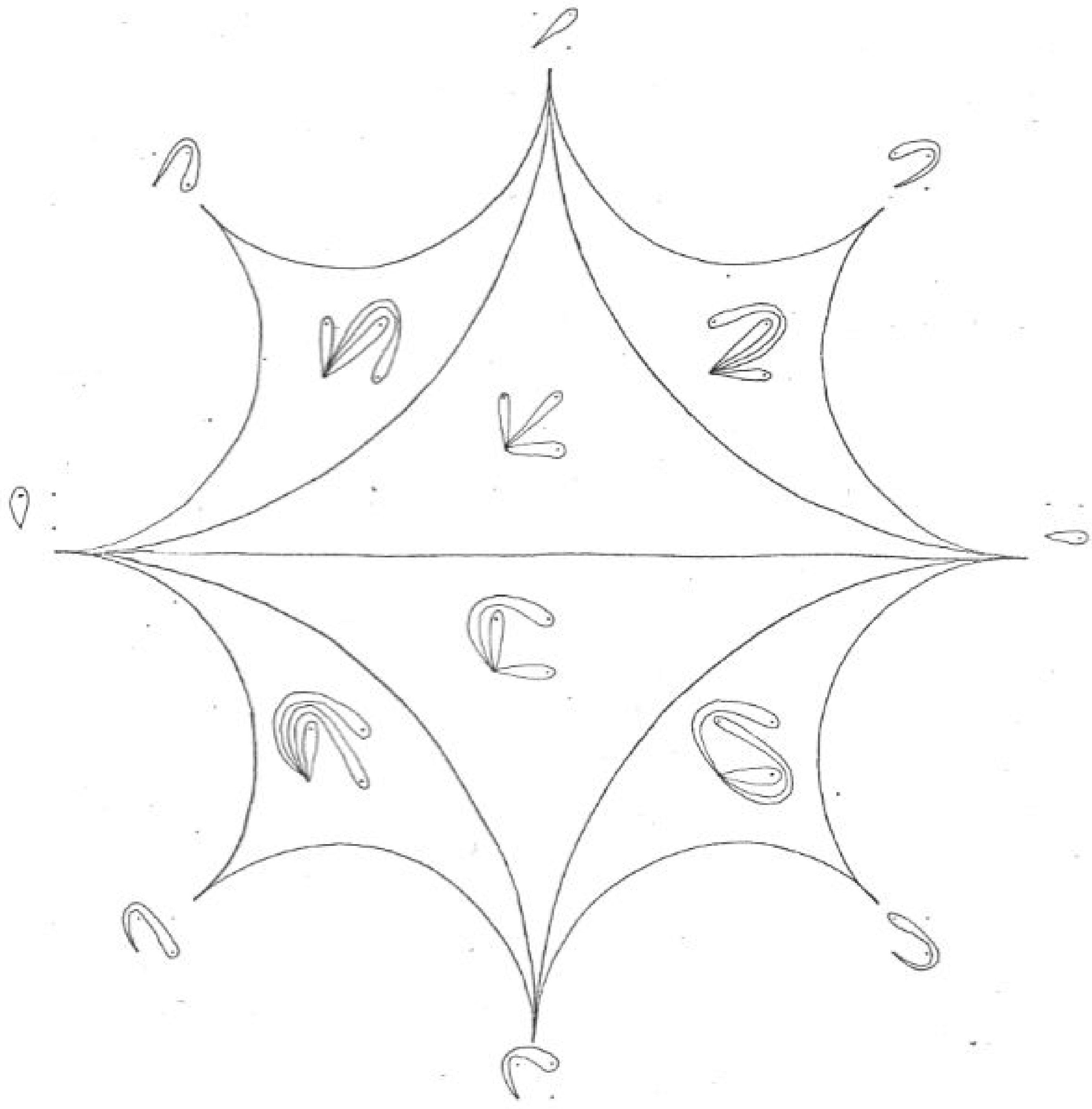
[HT] Hatcher, A. and Thurston, W., "Rational knots: incompressible surfaces and Dehn surgery", to appear.

[T] Thurston, W., The Geometry and Topology of 3-Manifolds, to appear.

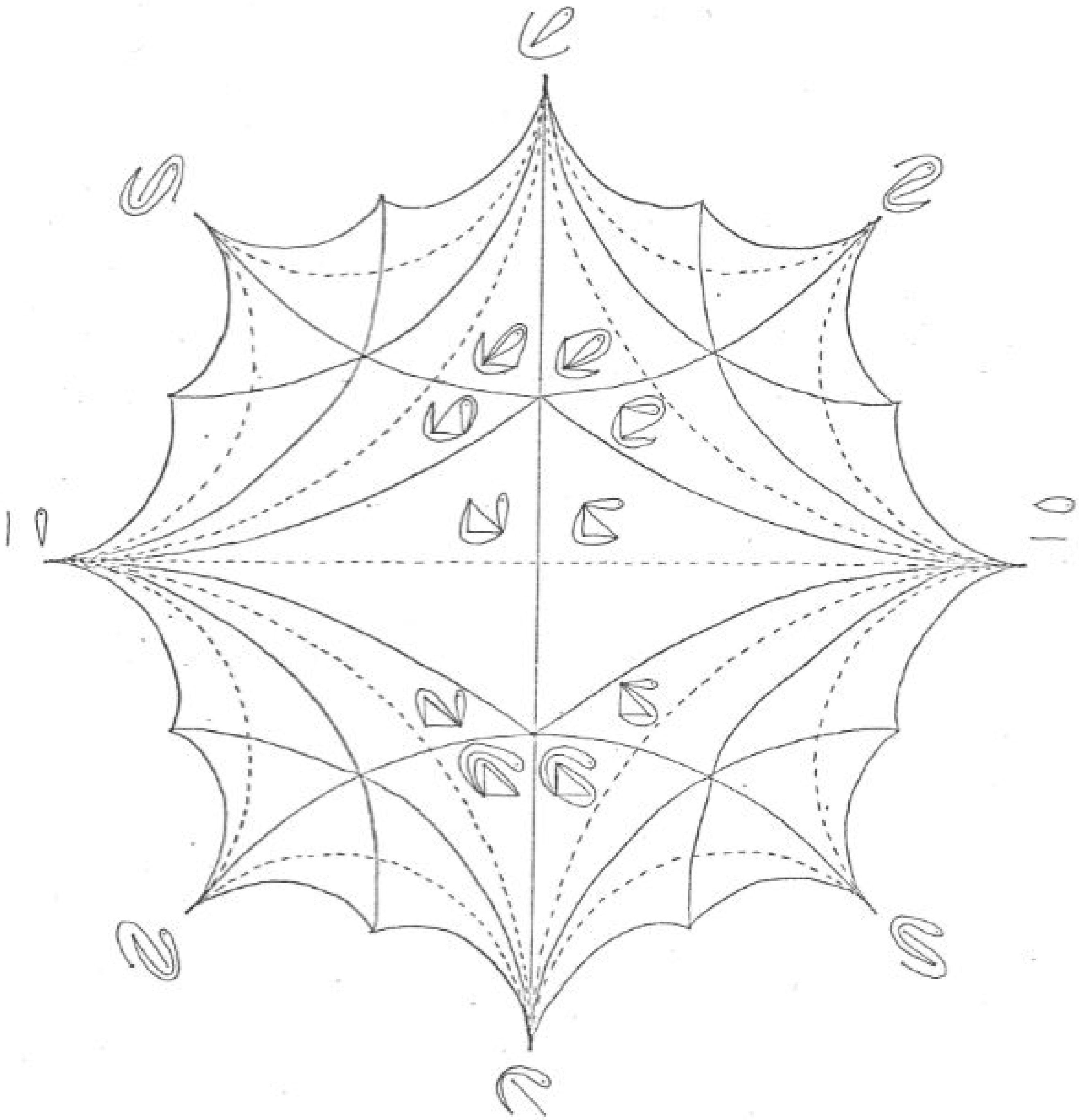
$\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ - A



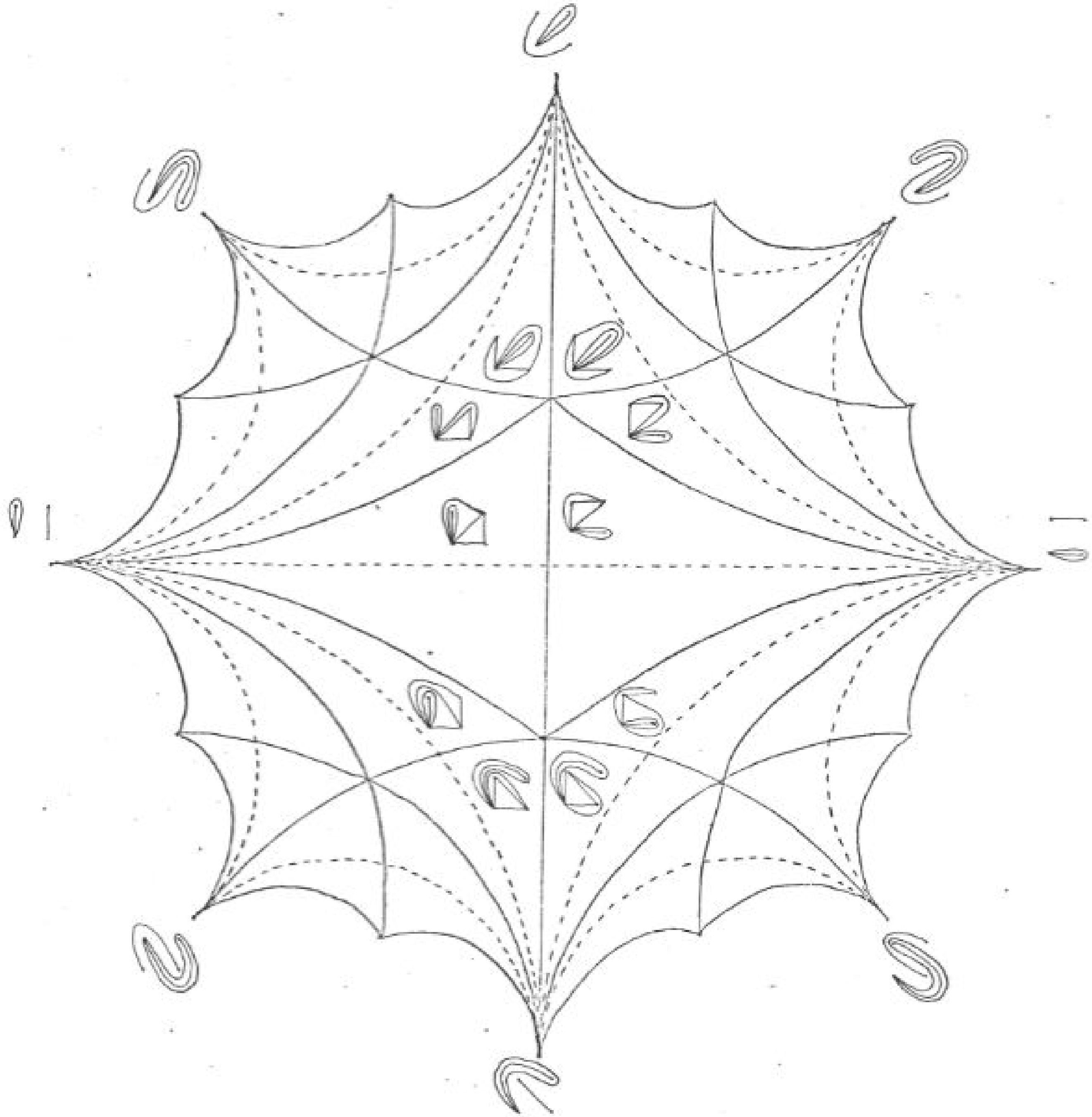
$\begin{bmatrix}0 & 0\\ a & 0\end{bmatrix}$ – A′



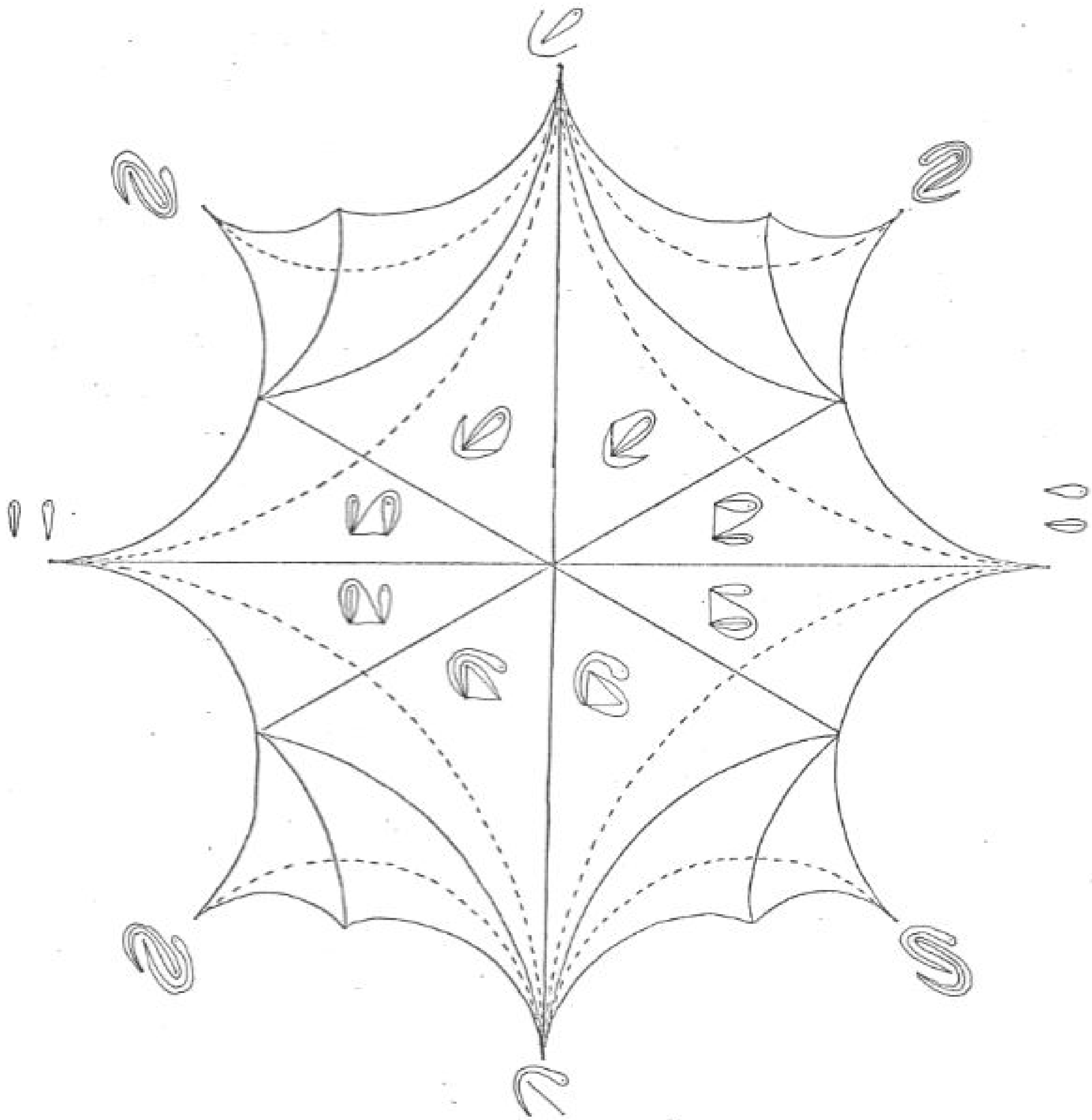
$\begin{bmatrix} a & 0 \\ a & a \end{bmatrix}$ - B



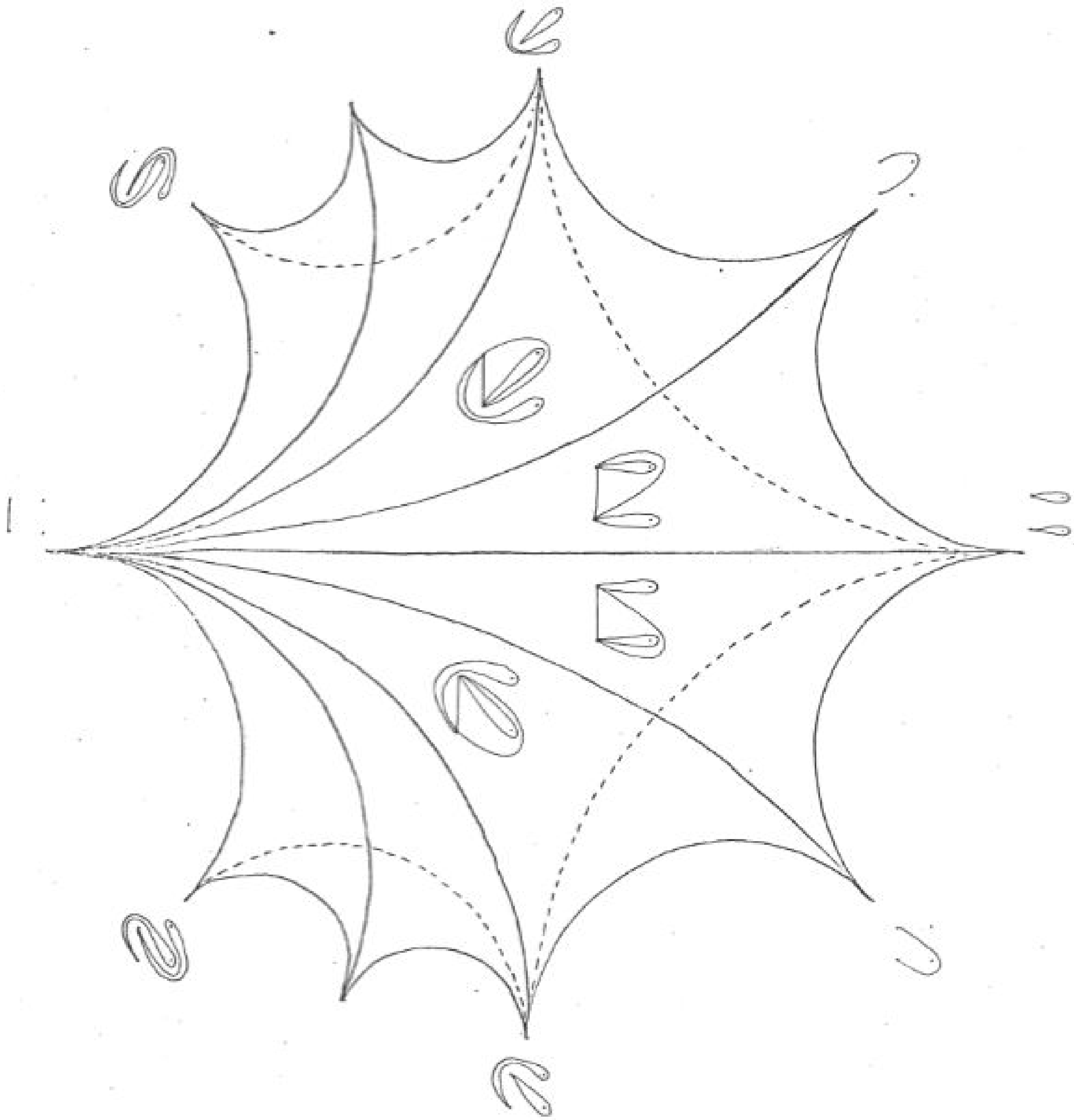
$$\begin{bmatrix} a & a \\ 3a & a \end{bmatrix} - B'$$



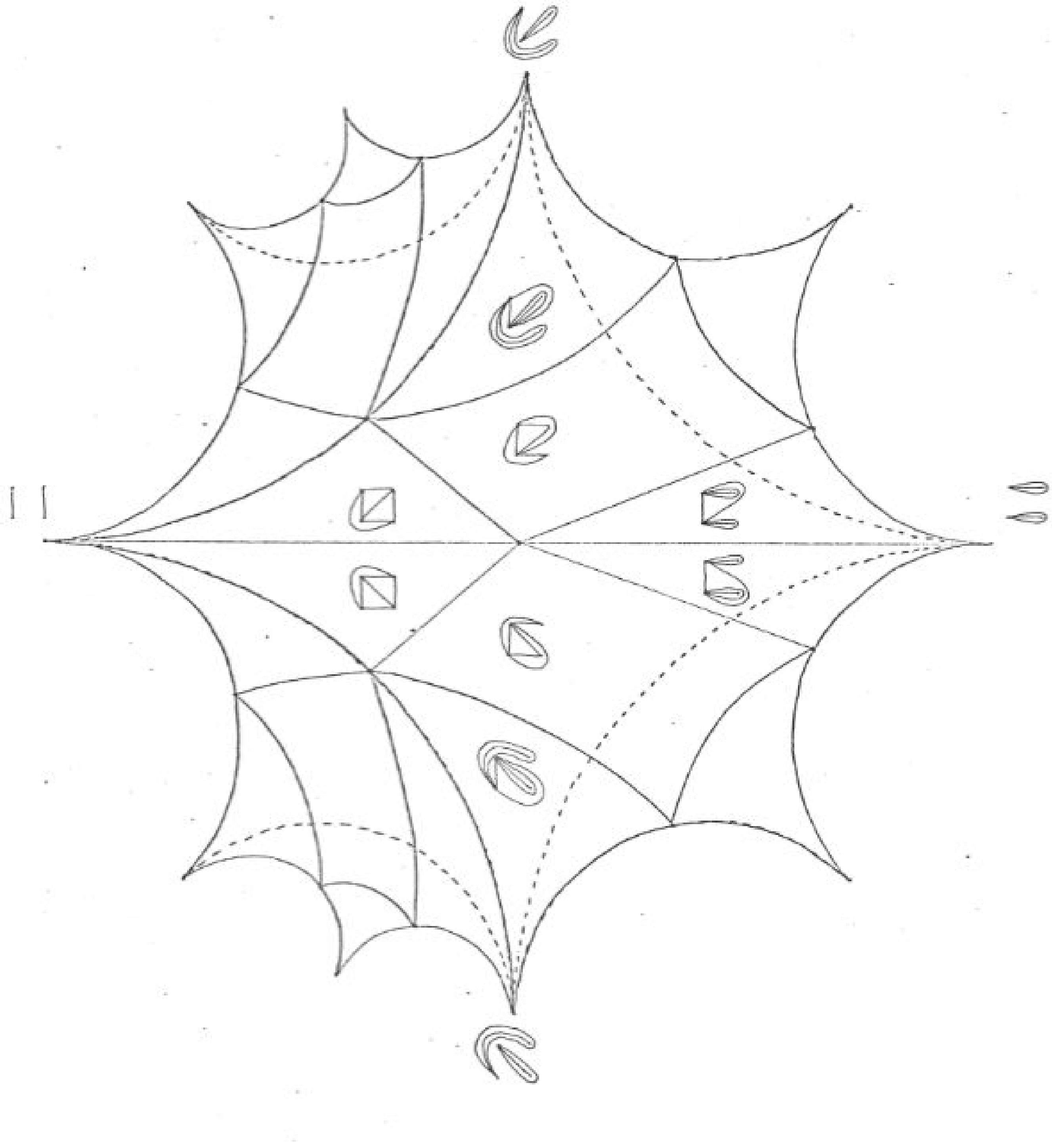
$\begin{bmatrix} a & 0 \\ 2a & a \end{bmatrix}$ - C



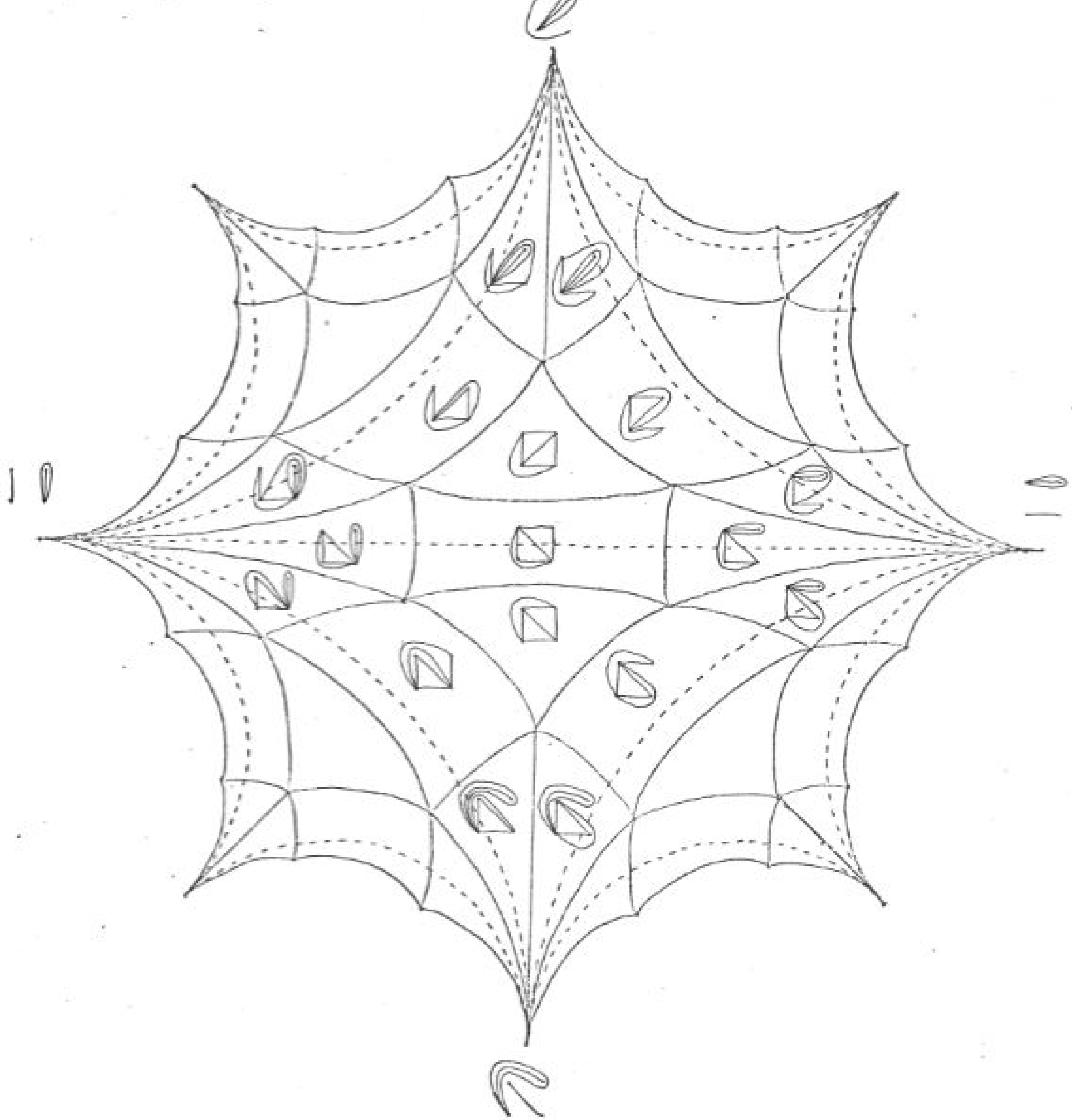
$\begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}$ - D



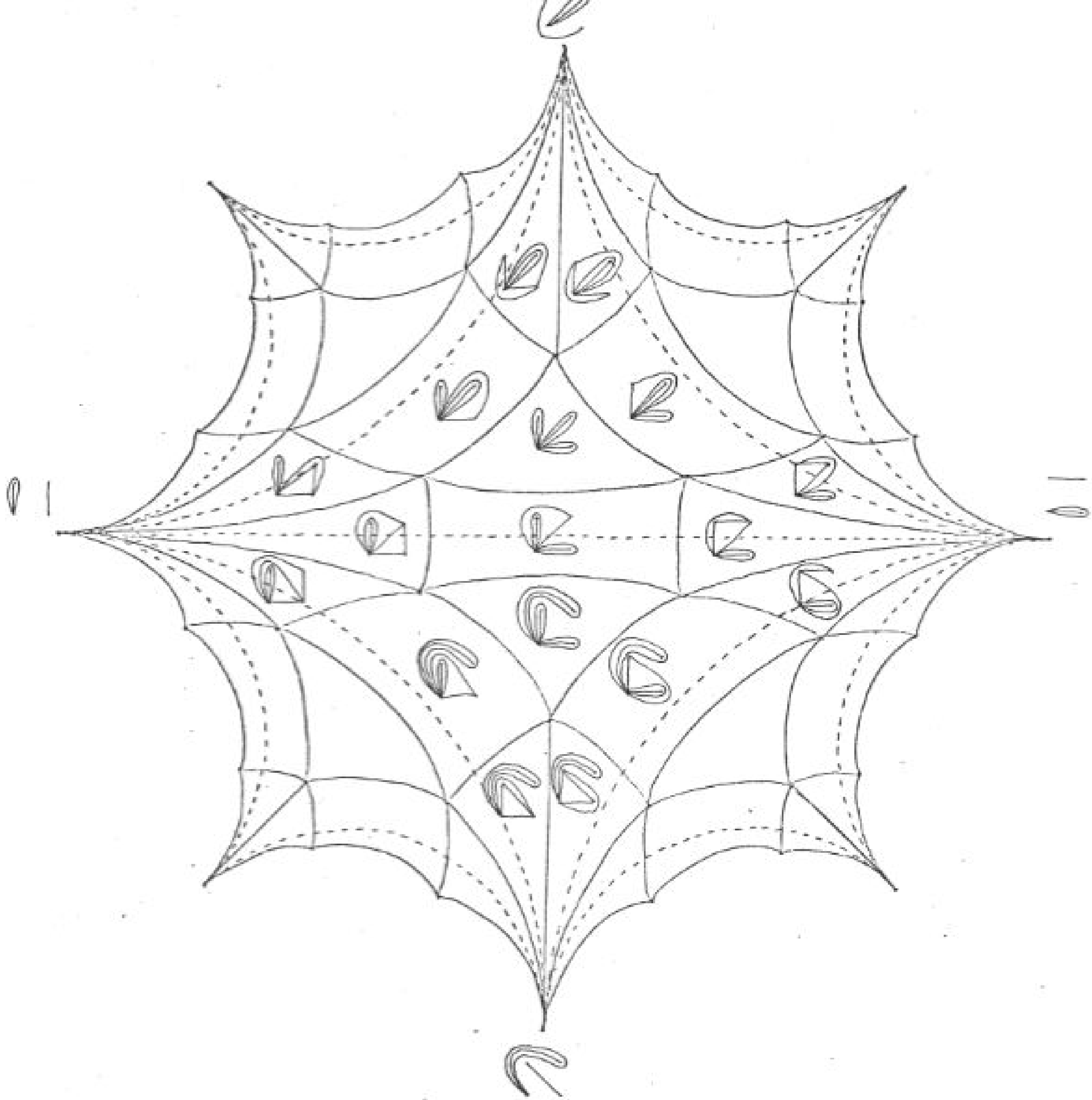
$\begin{bmatrix} a+b & a \\ a+b & a \end{bmatrix}$ - AD



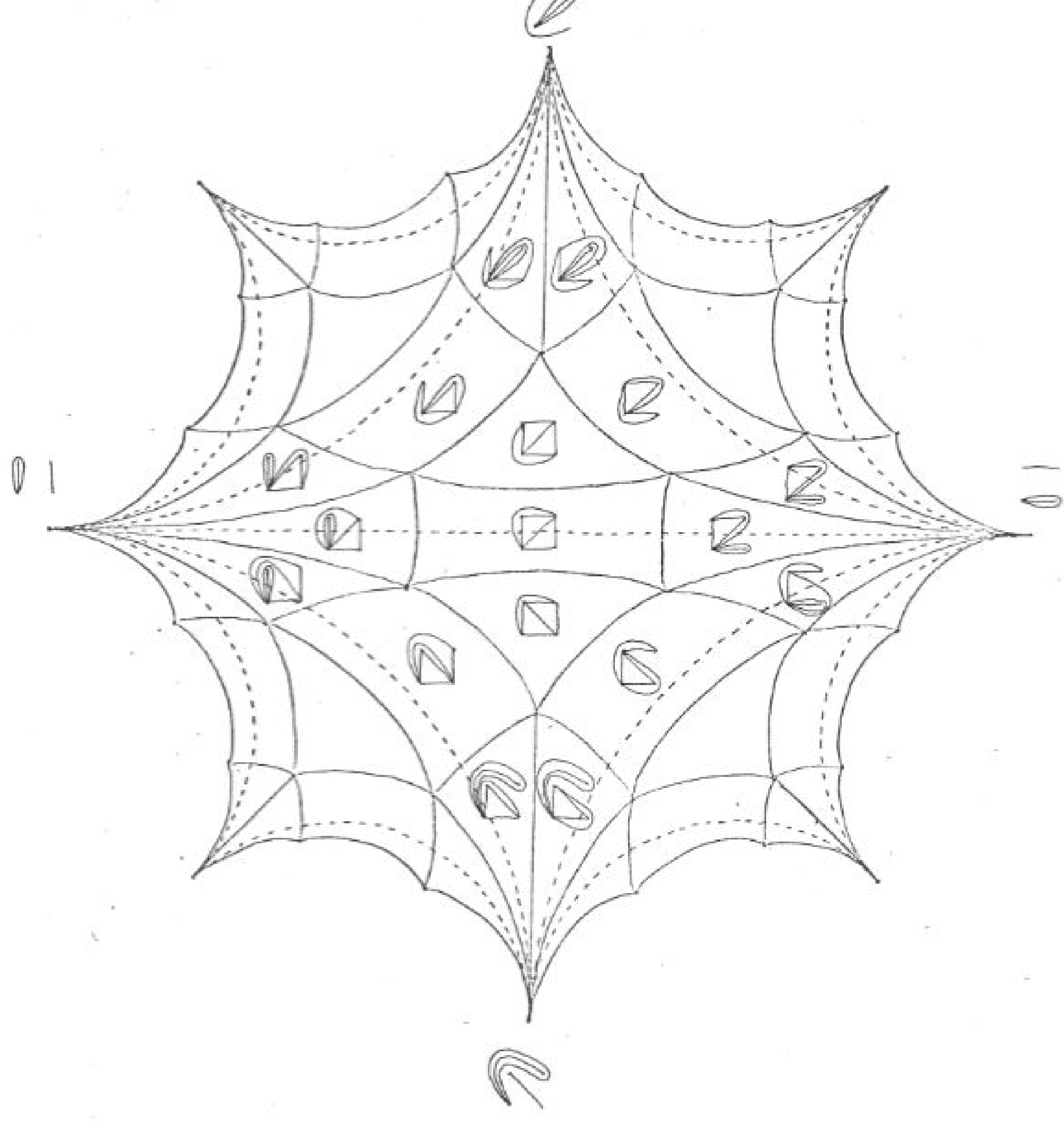
$\begin{bmatrix} a+b & a \\ a+b & a+b \end{bmatrix}$ - AB



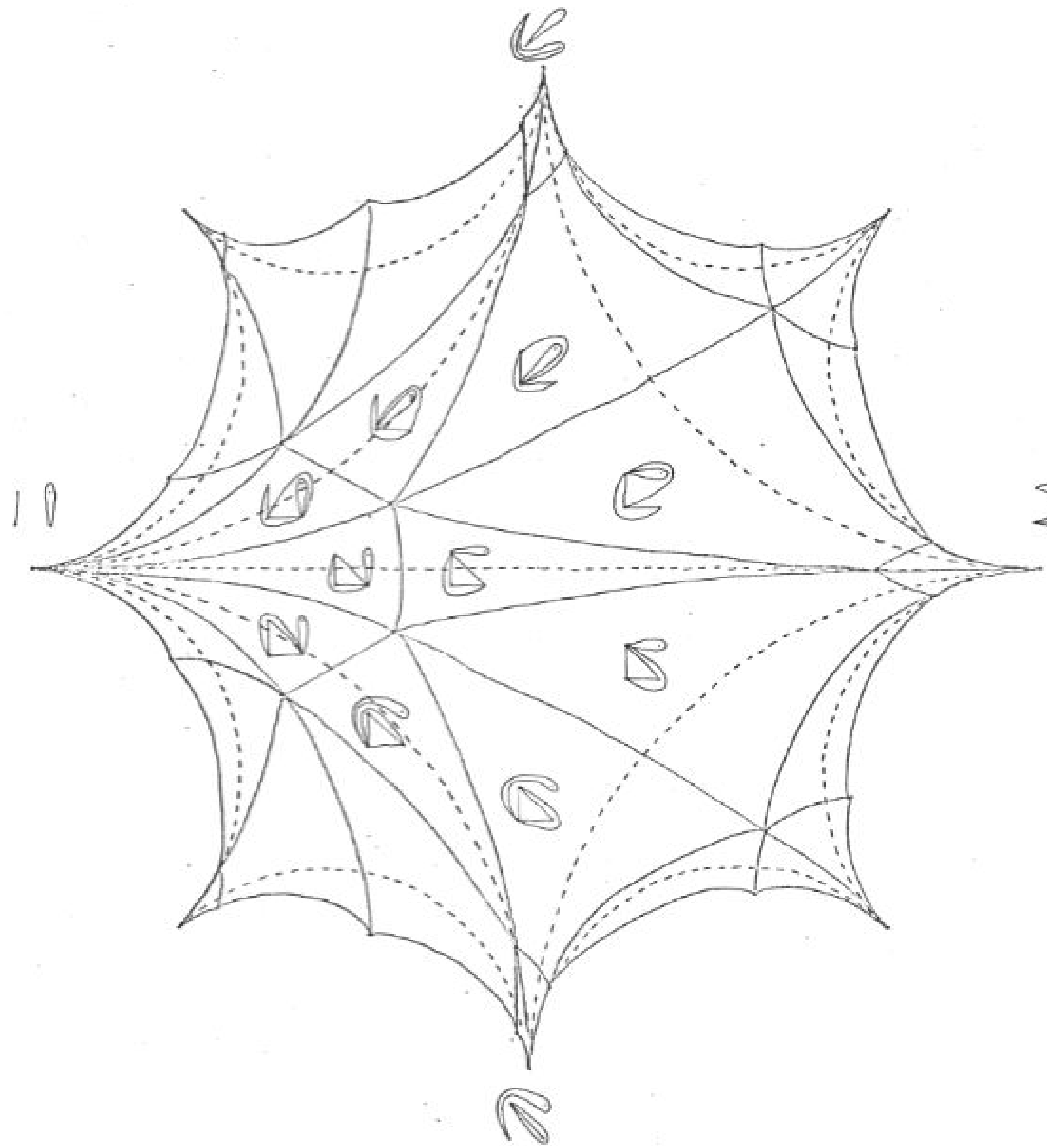
$\begin{bmatrix} a & a \\ 3a+b & a \end{bmatrix}$ - A′B′



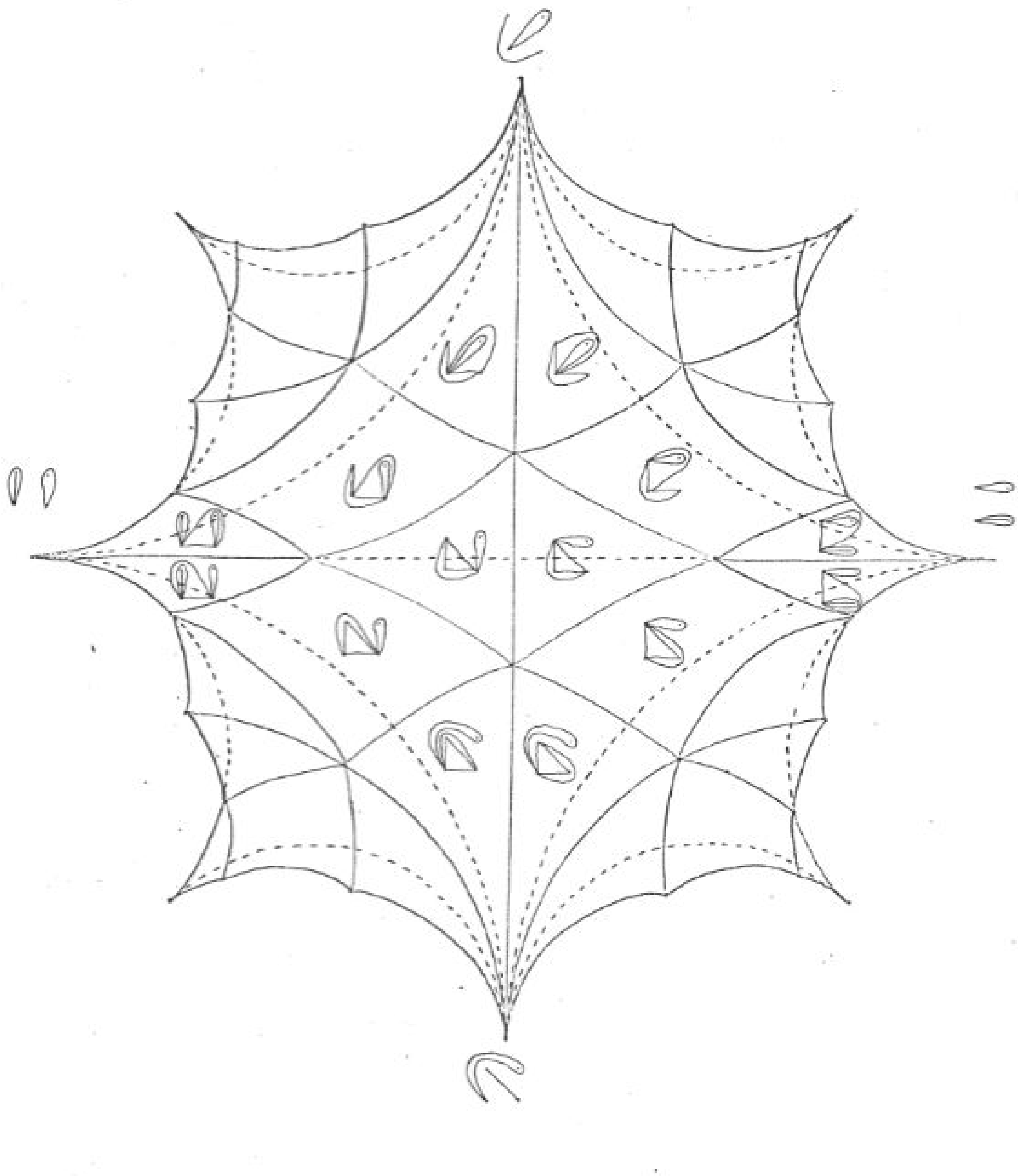
$\begin{bmatrix} a+b & a+b \\ a+3b & a+b \end{bmatrix}$ - AB′



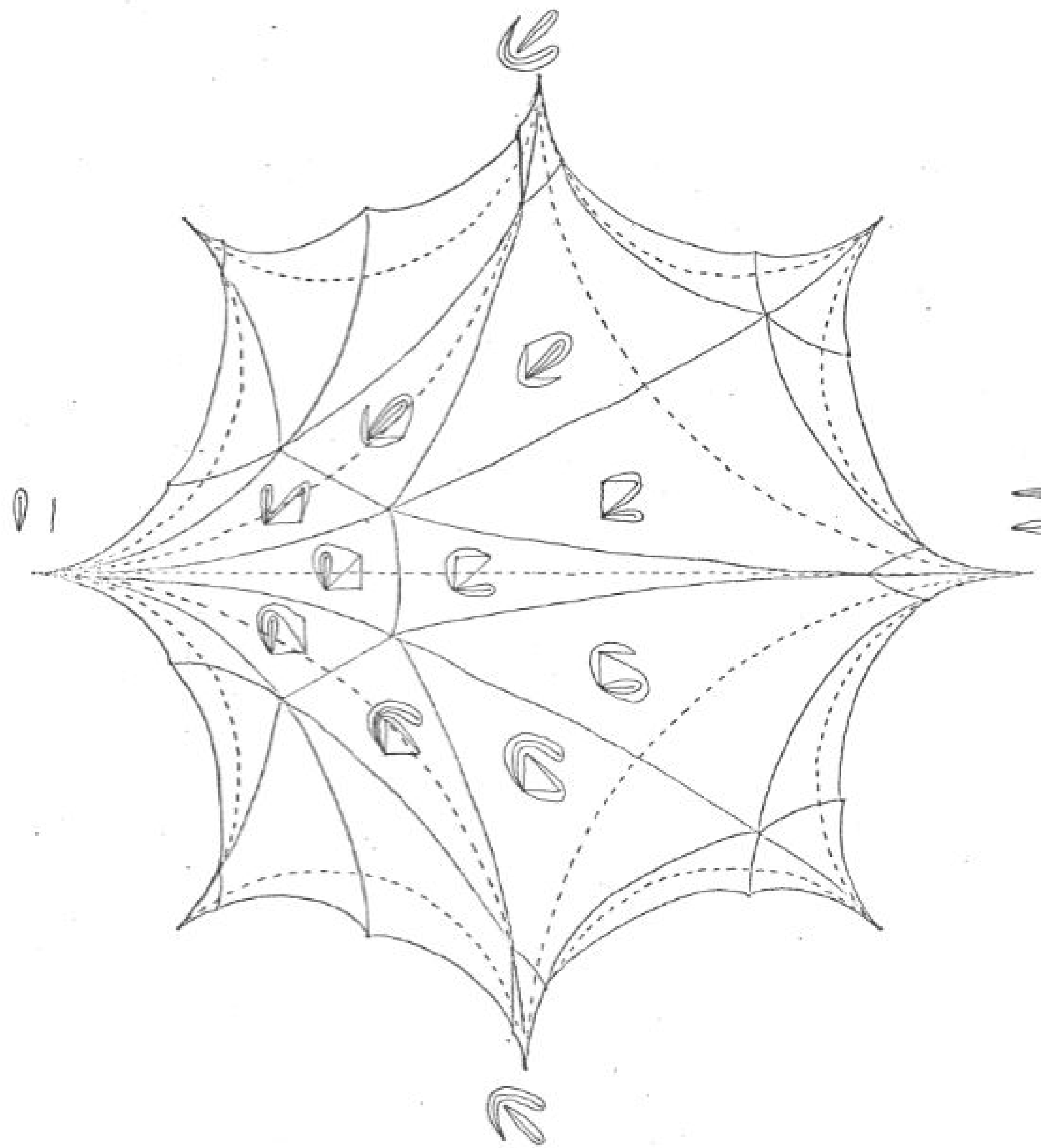
$\begin{bmatrix} a+b & 0 \\ a+b & a \end{bmatrix}$ - BD



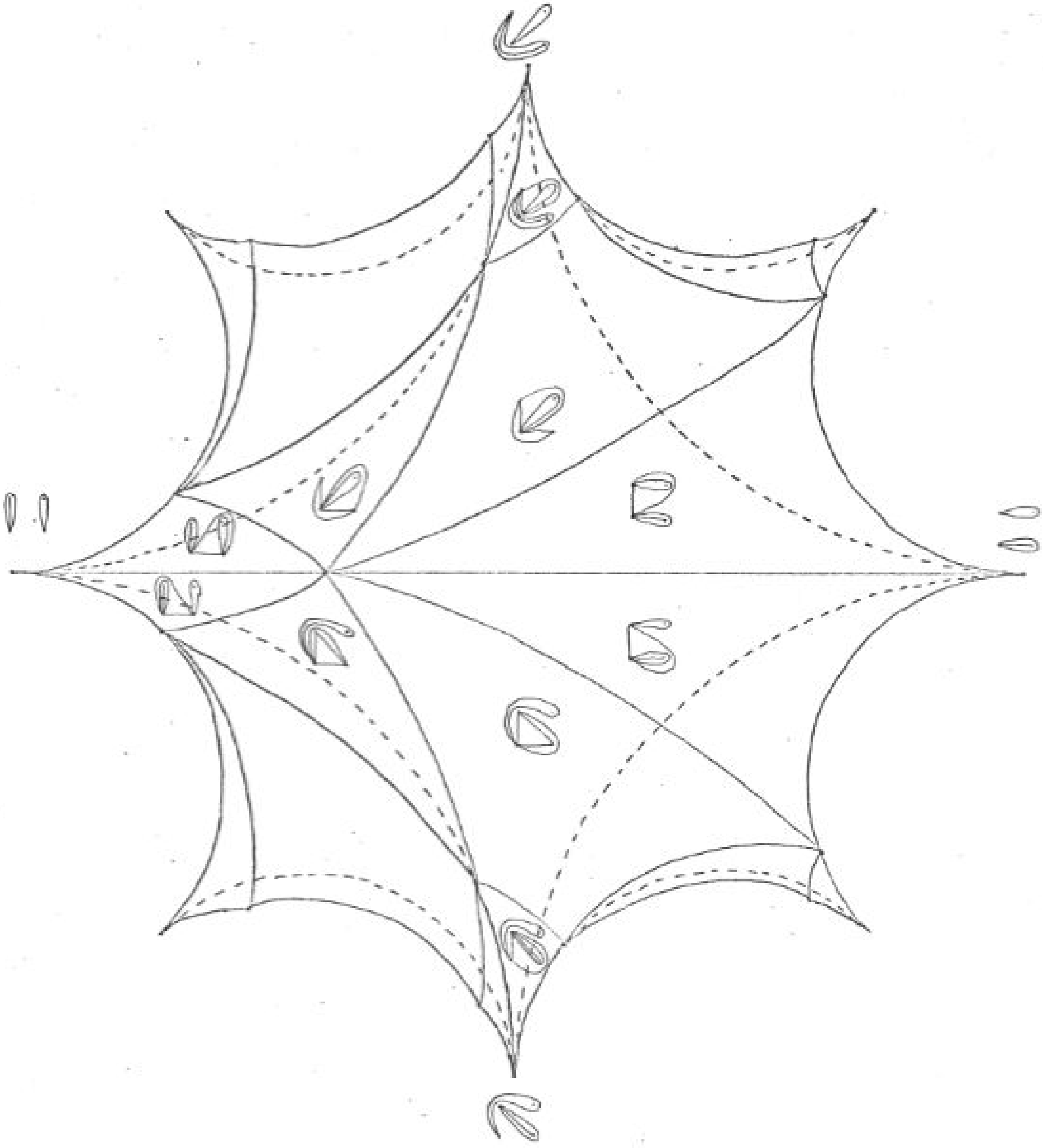
$$\begin{bmatrix} a+b & 0 \\ a+2b & a+b \end{bmatrix} - BC$$



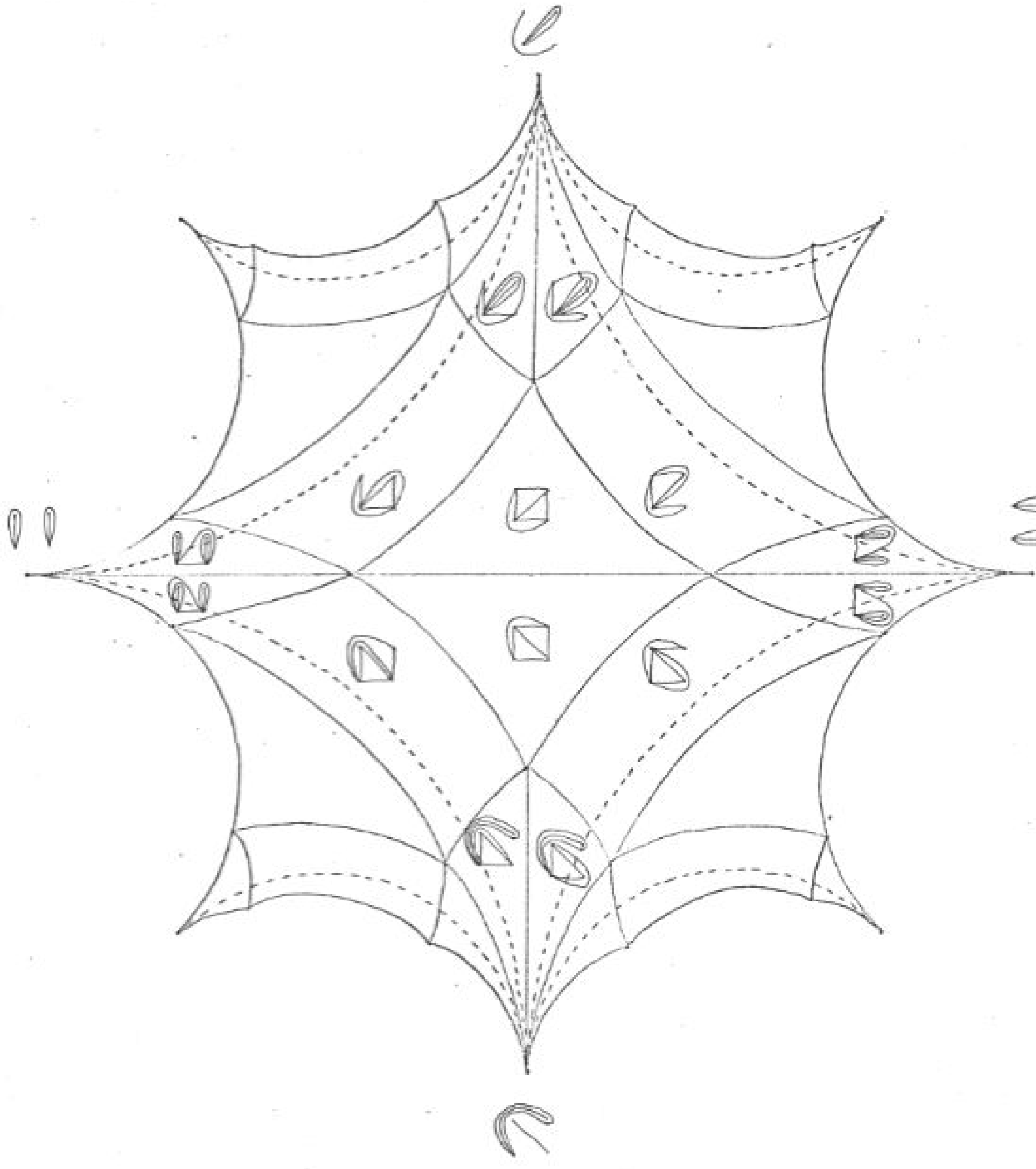
$\begin{bmatrix} a+b & a \\ 3a+b & a \end{bmatrix}$ - B'D



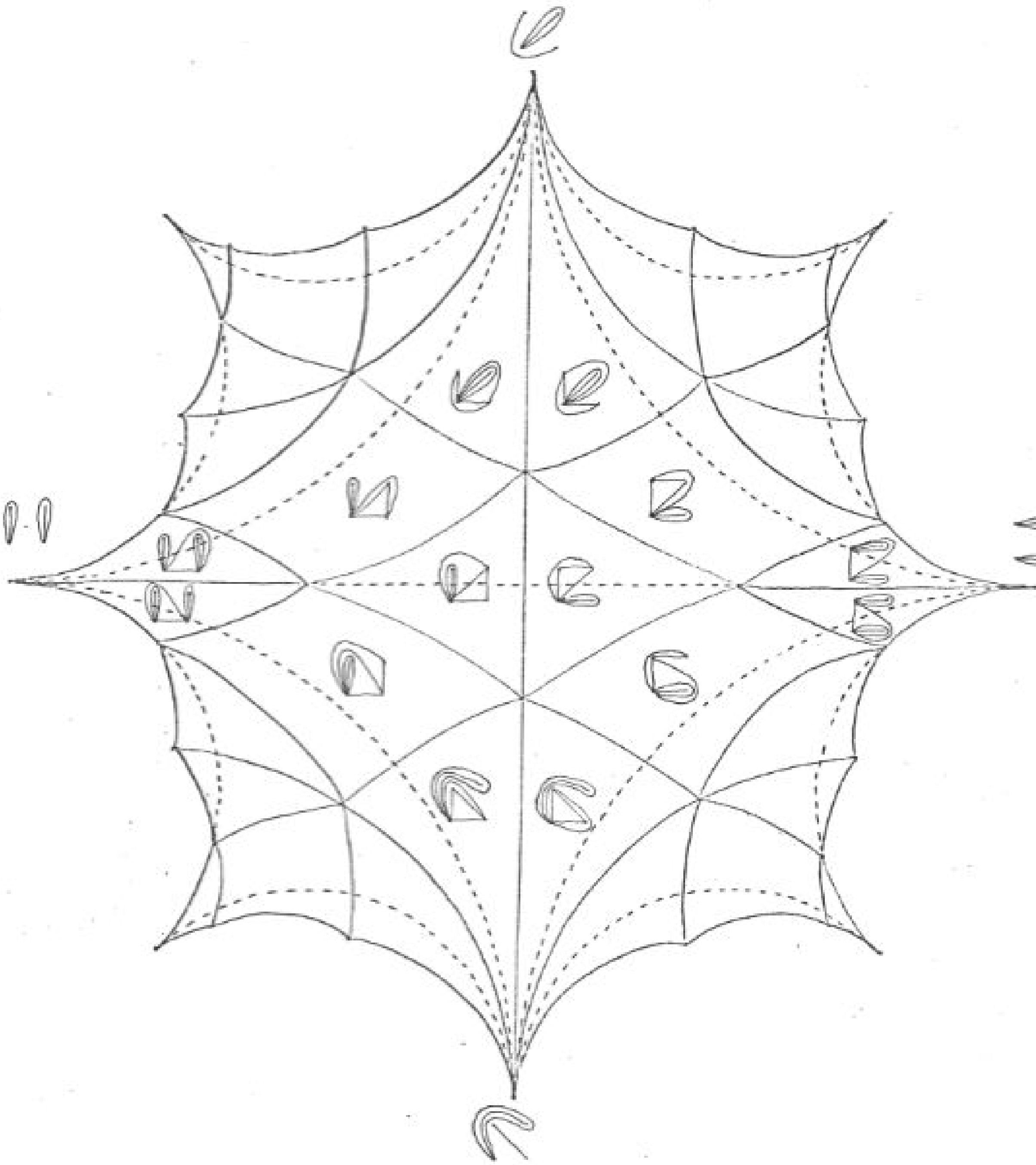
$$\begin{bmatrix} a+b & 0 \\ 2a+b & a \end{bmatrix} - CD$$



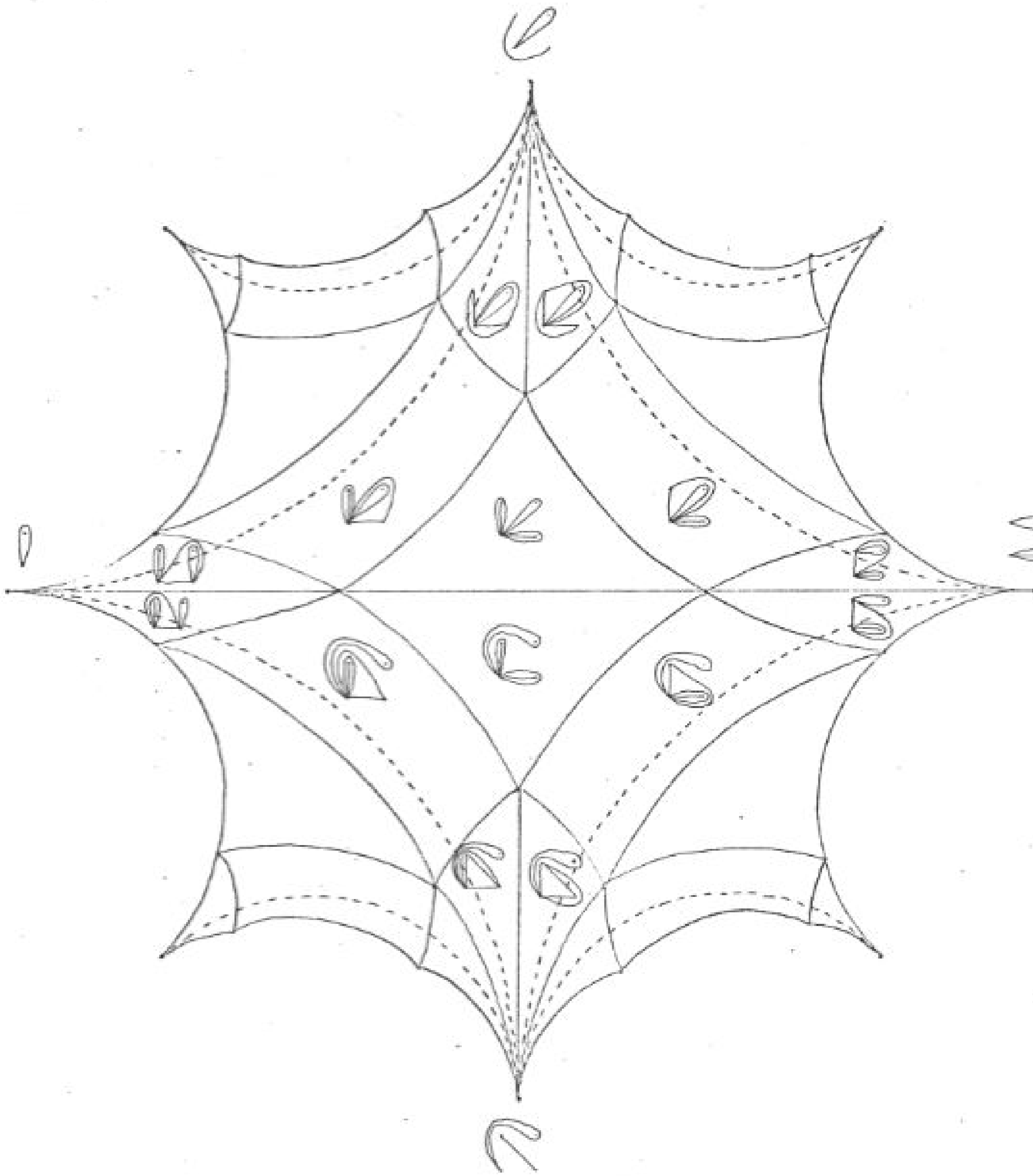
$$\begin{bmatrix} a+b & a \\ a+2b & a+b \end{bmatrix} - AC$$



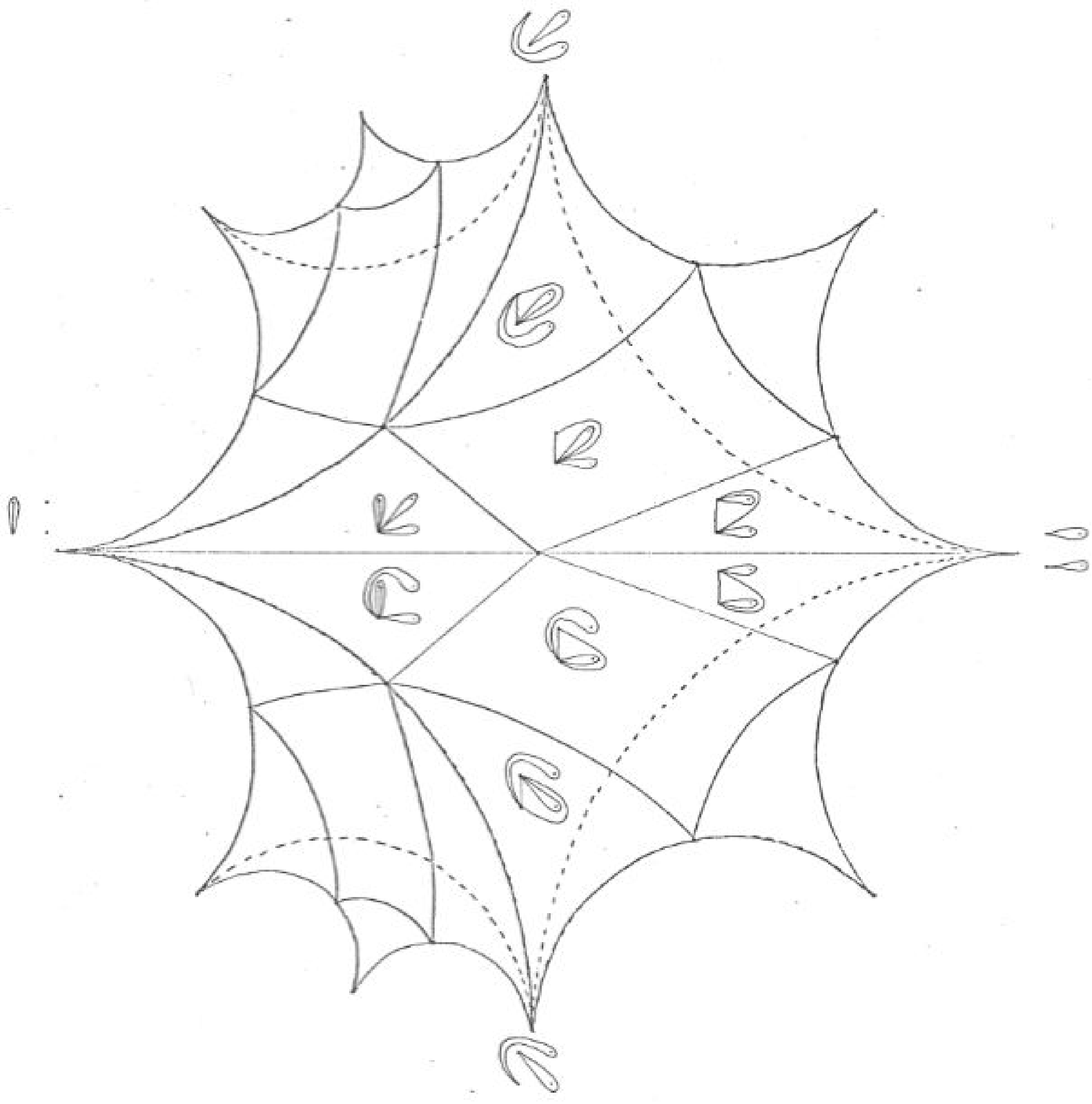
$$\begin{bmatrix} a+b & a \\ 3a+2b & a+b \end{bmatrix} - B'C$$



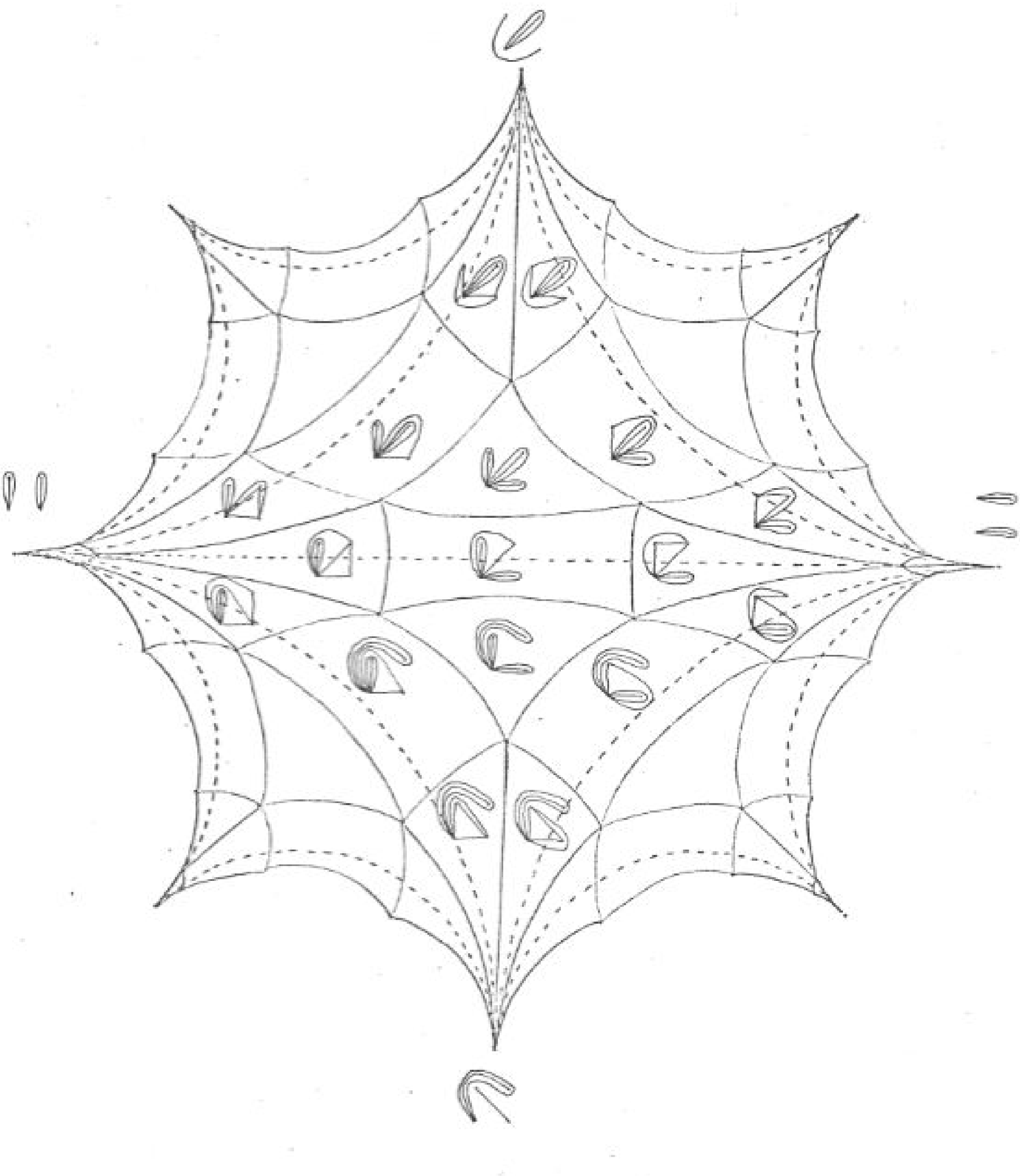
$$\begin{bmatrix} a & 0 \\ 2a+b & a \end{bmatrix} - A'C$$



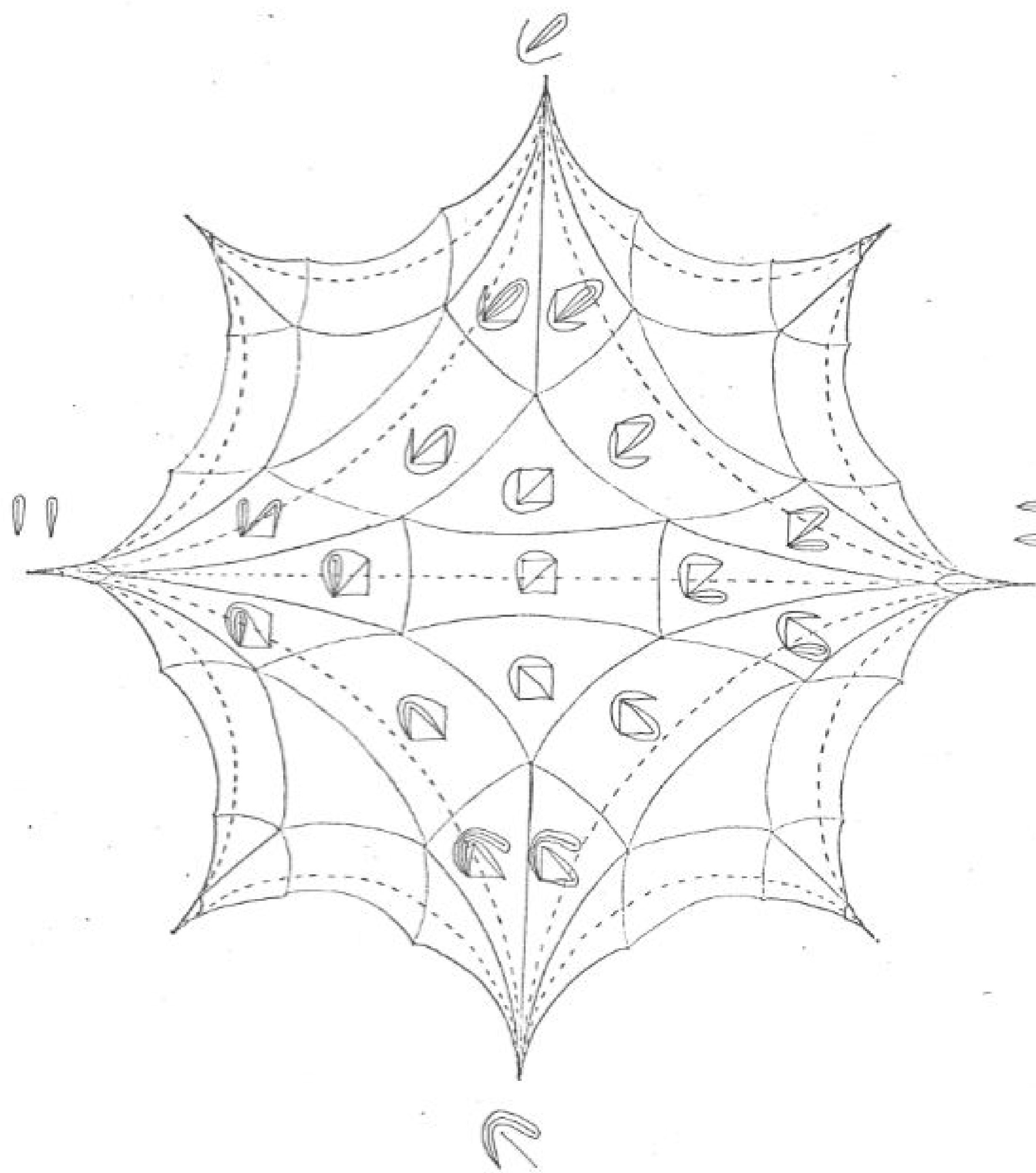
$\begin{bmatrix} a & 0 \\ a+b & 0 \end{bmatrix}$ - A′D



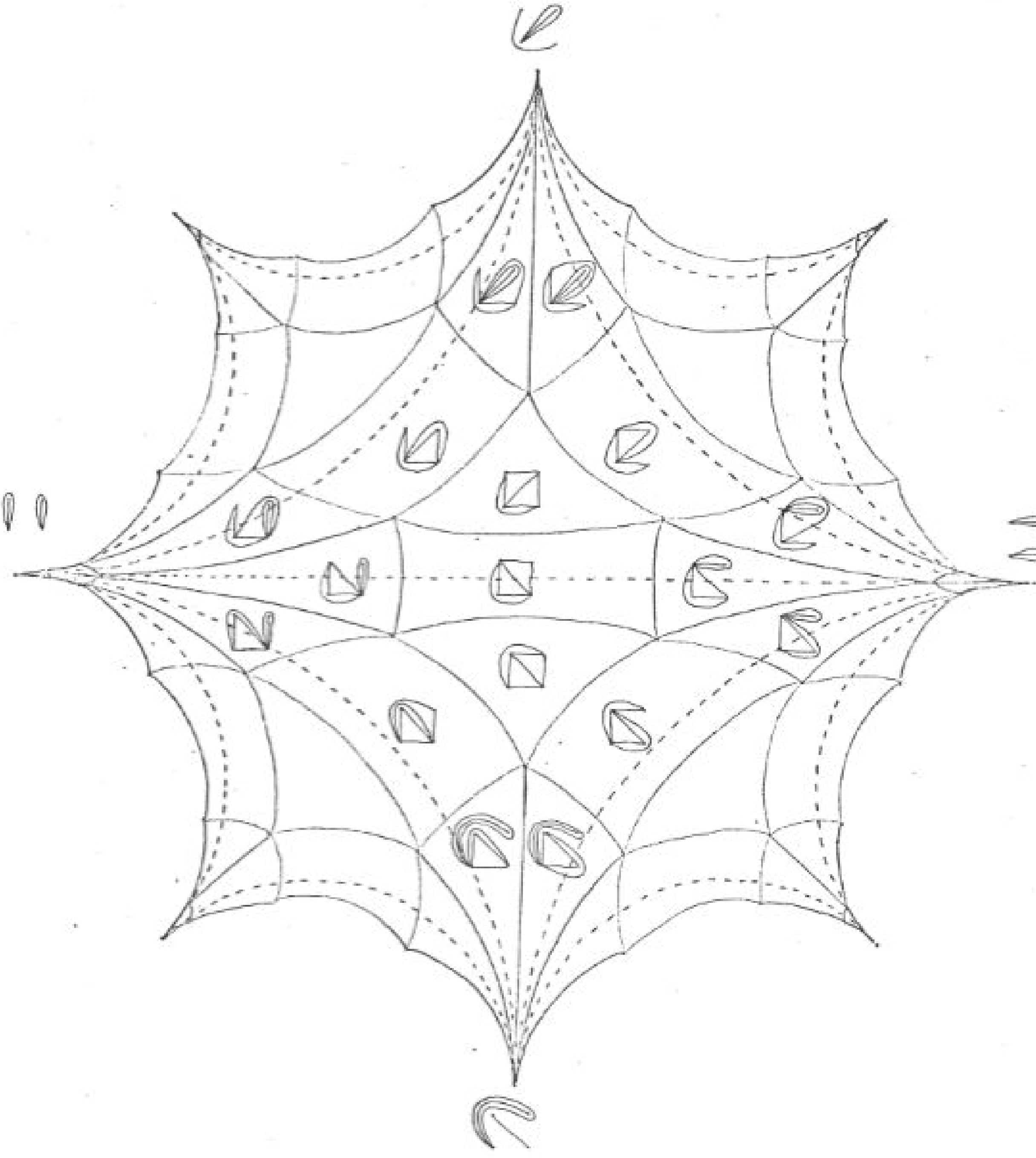
$$\begin{bmatrix} a+b & a \\ 3a+2b+c & a+b \end{bmatrix} - A'B'C$$



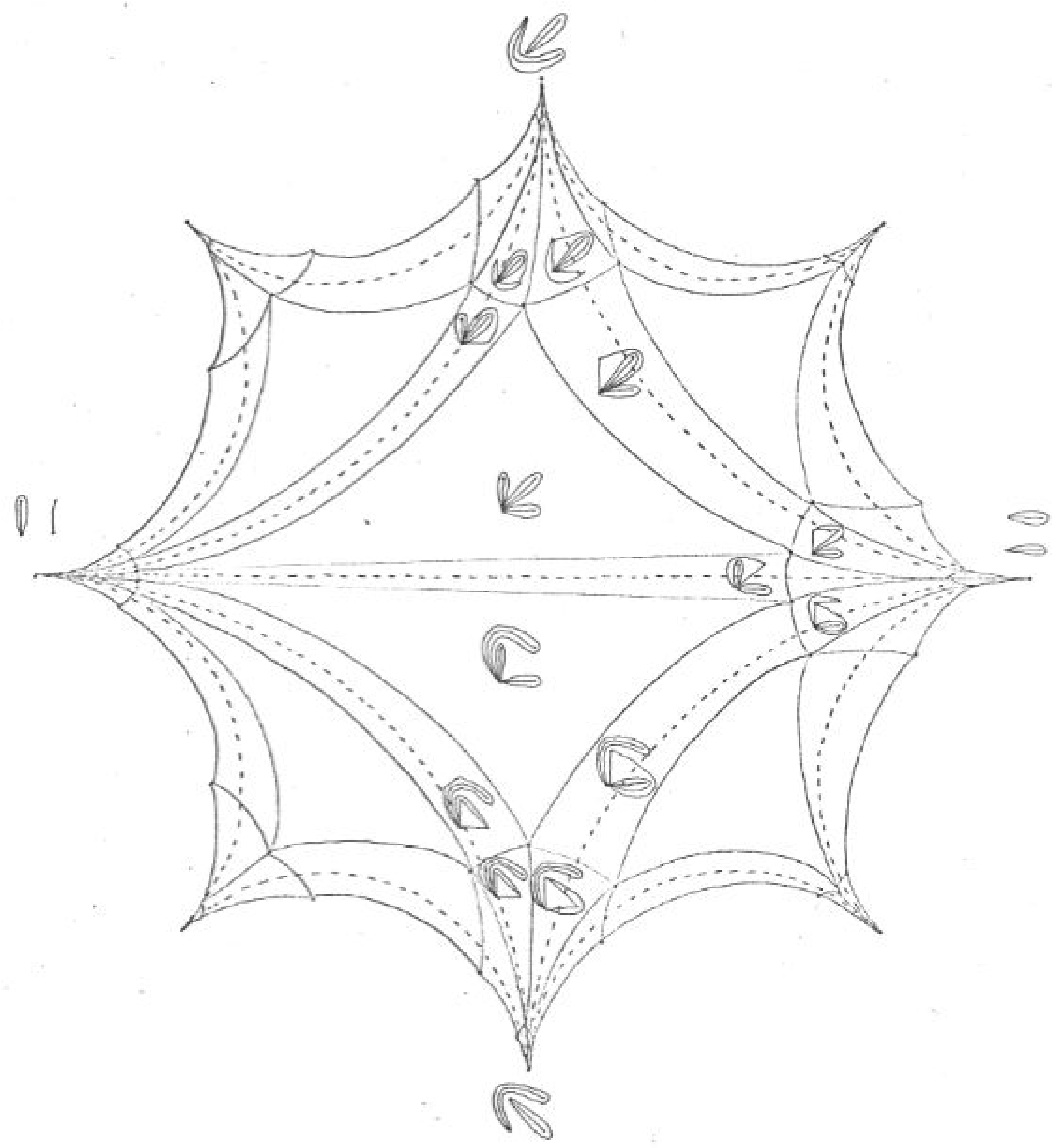
$$\begin{bmatrix} a+b+c & a+b \\ a+3b+2c & a+b+c \end{bmatrix} - AB'C$$



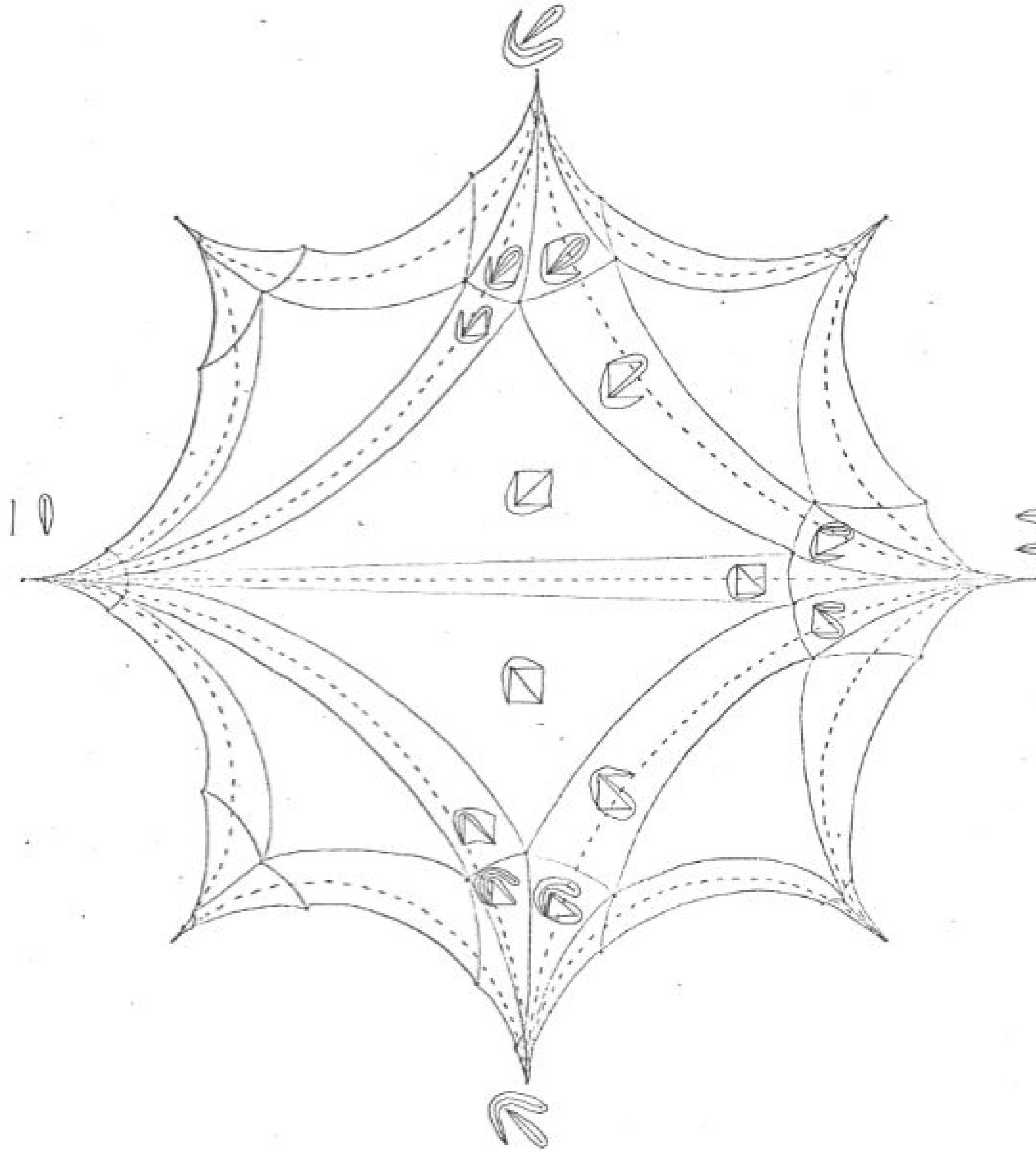
$$\begin{bmatrix} a+b+c & a \\ a+b+2c & a+b+c \end{bmatrix} - ABC$$



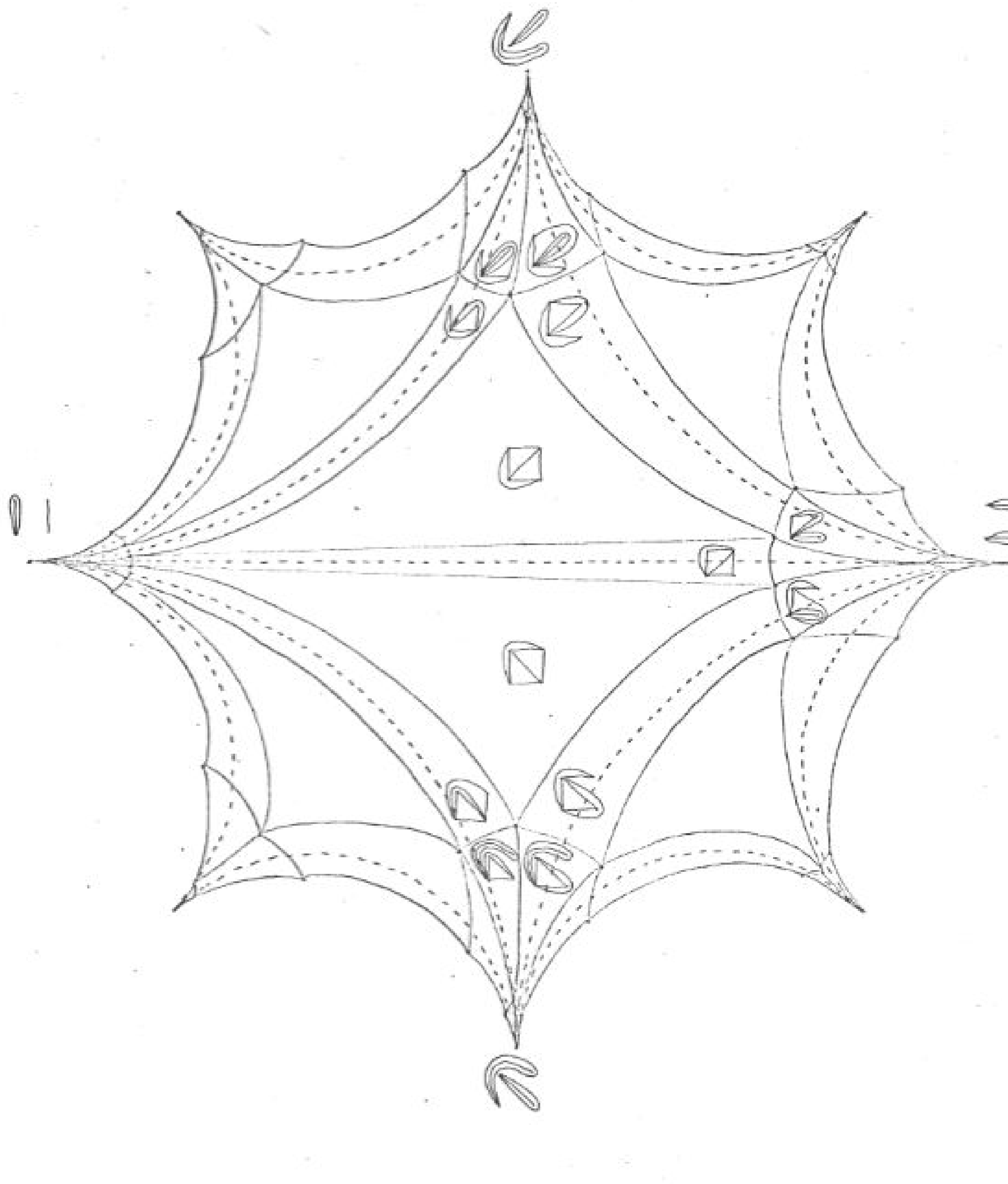
$\begin{bmatrix} a+b & a \\ 3a+b+c & a \end{bmatrix}$ - A´B´D



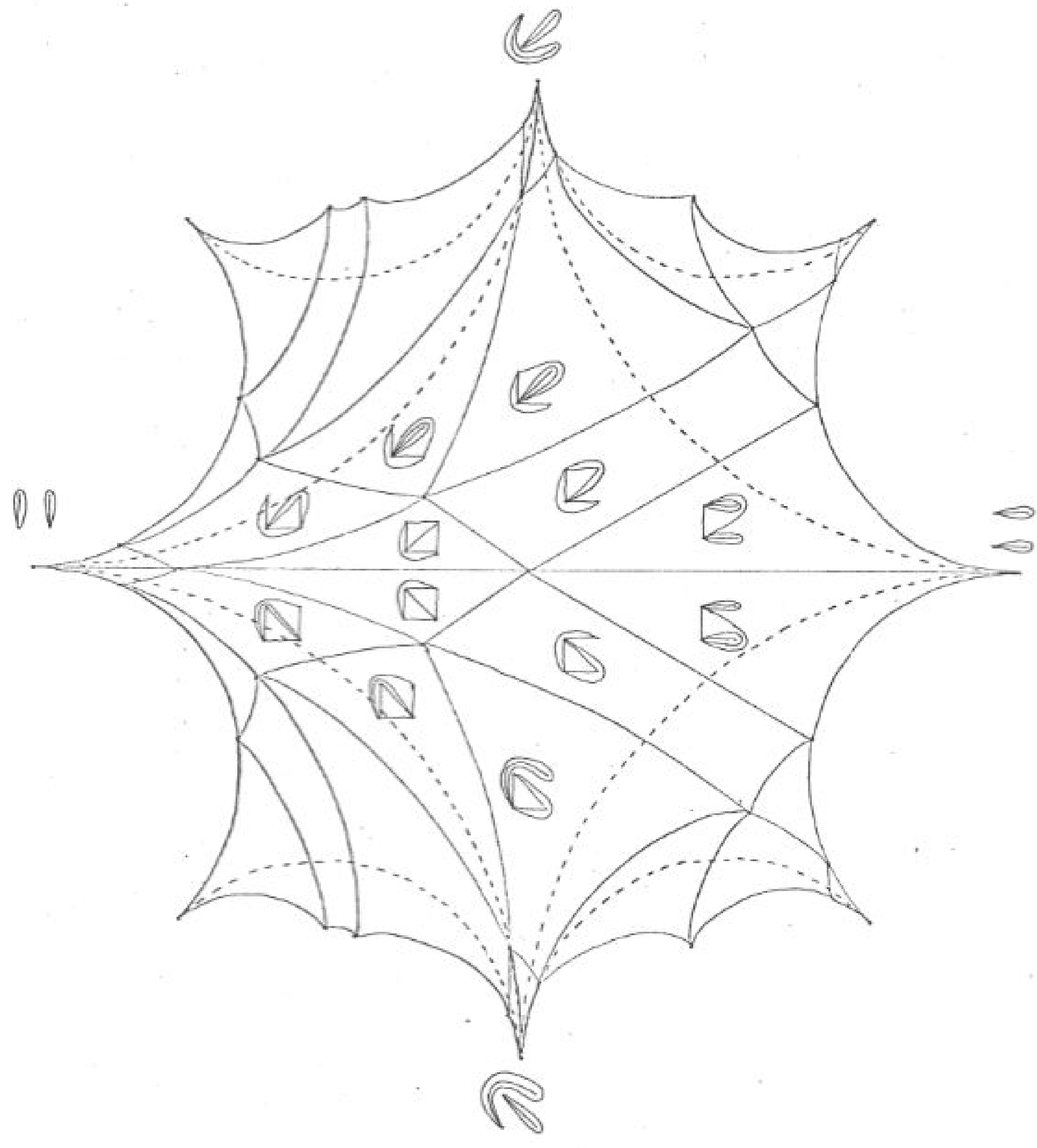
$\begin{bmatrix} a+b+c & a \\ a+b+c & a+b \end{bmatrix}$ - ABD



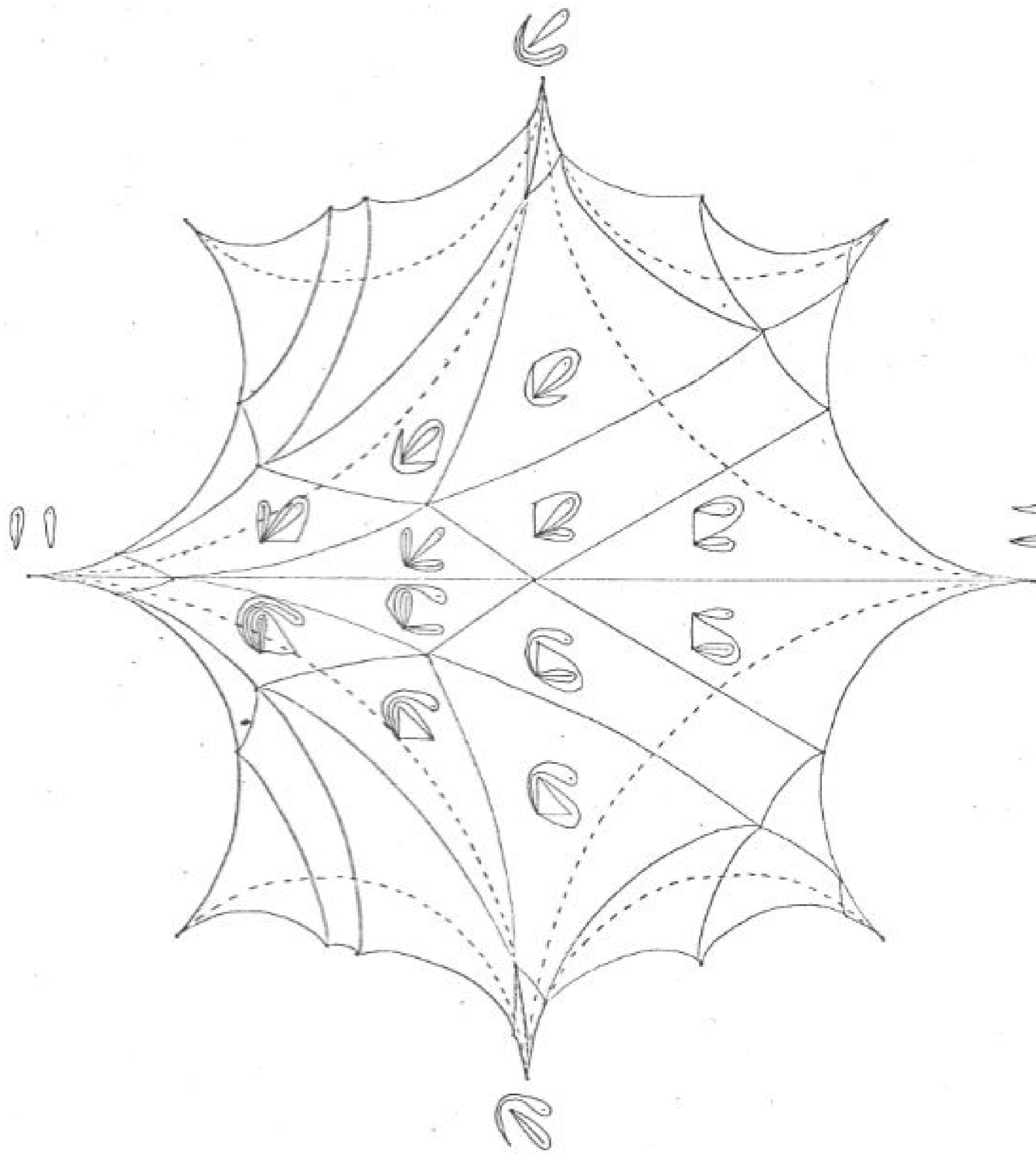
$\begin{bmatrix} a+b+c & a+b \\ 3a+b+c & a+b \end{bmatrix}$ - AB′D



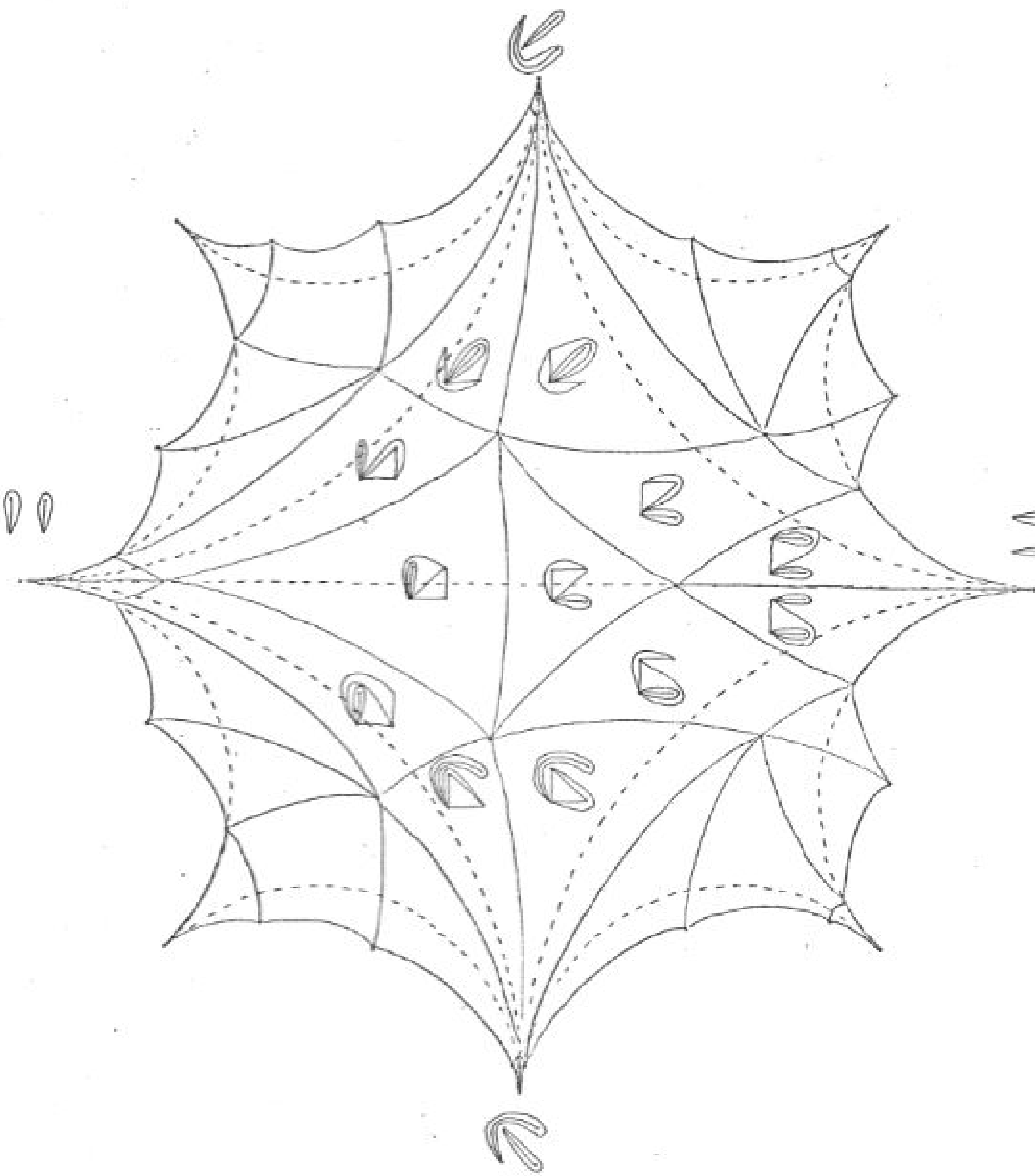
$$\begin{bmatrix} a+b+c & a \\ a+2b+c & a+b \end{bmatrix} - \text{ACD}$$



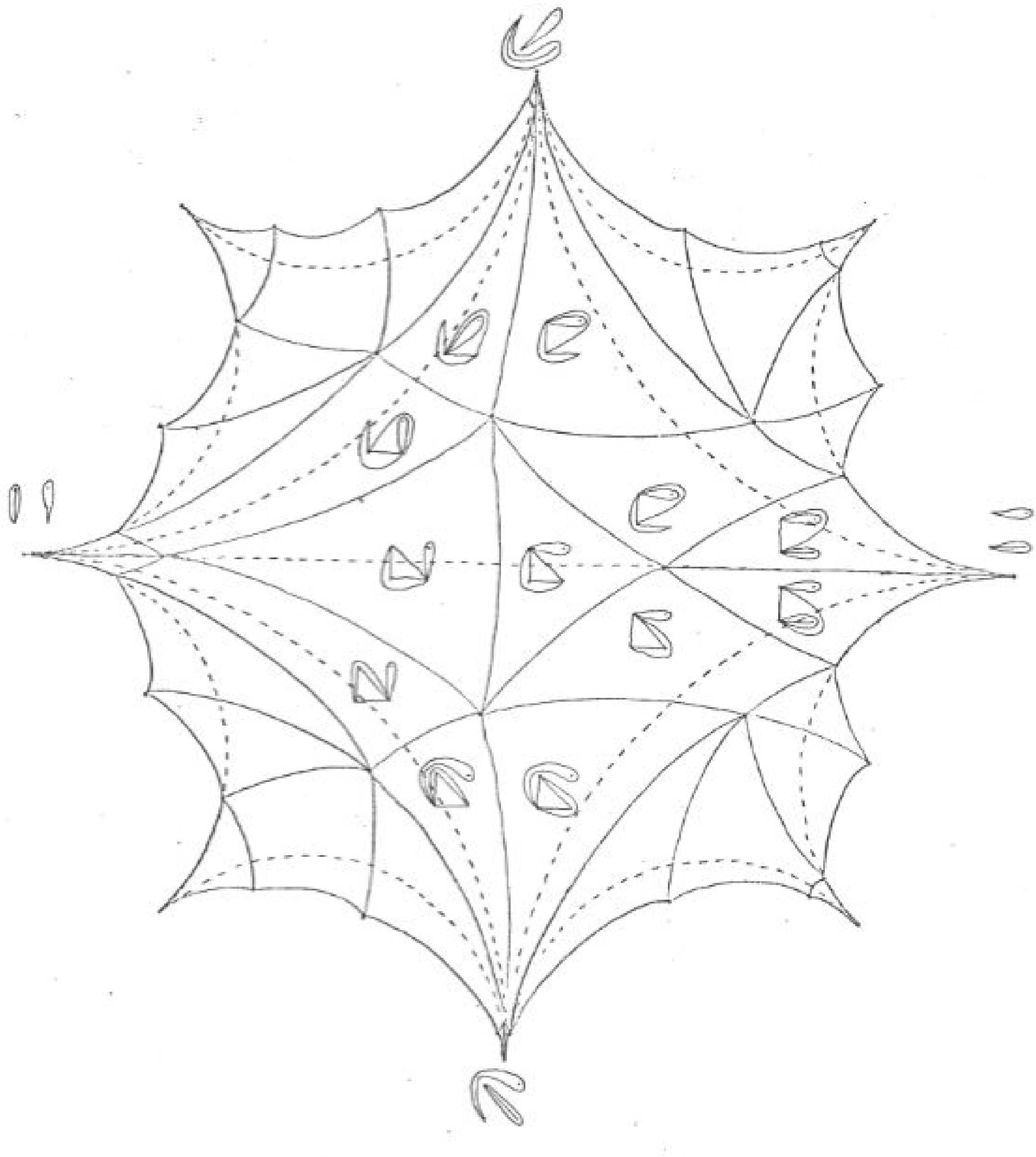
$$\begin{bmatrix} a+b & 0 \\ 2a+b+c & a \end{bmatrix} - A'CD$$



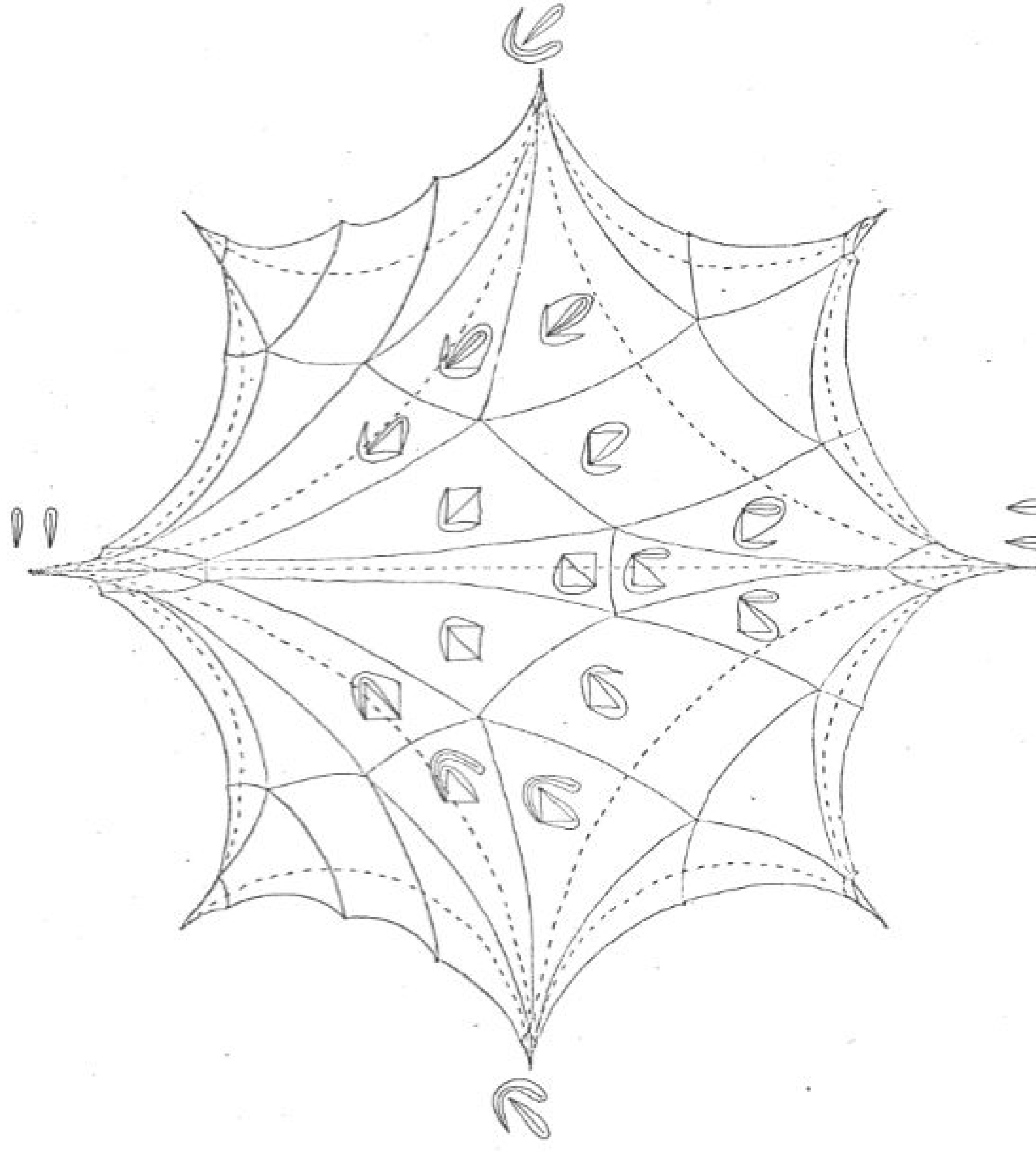
$$\begin{bmatrix} a+b+c & a \\ 3a+2b+c & a+b \end{bmatrix} - B'CD$$



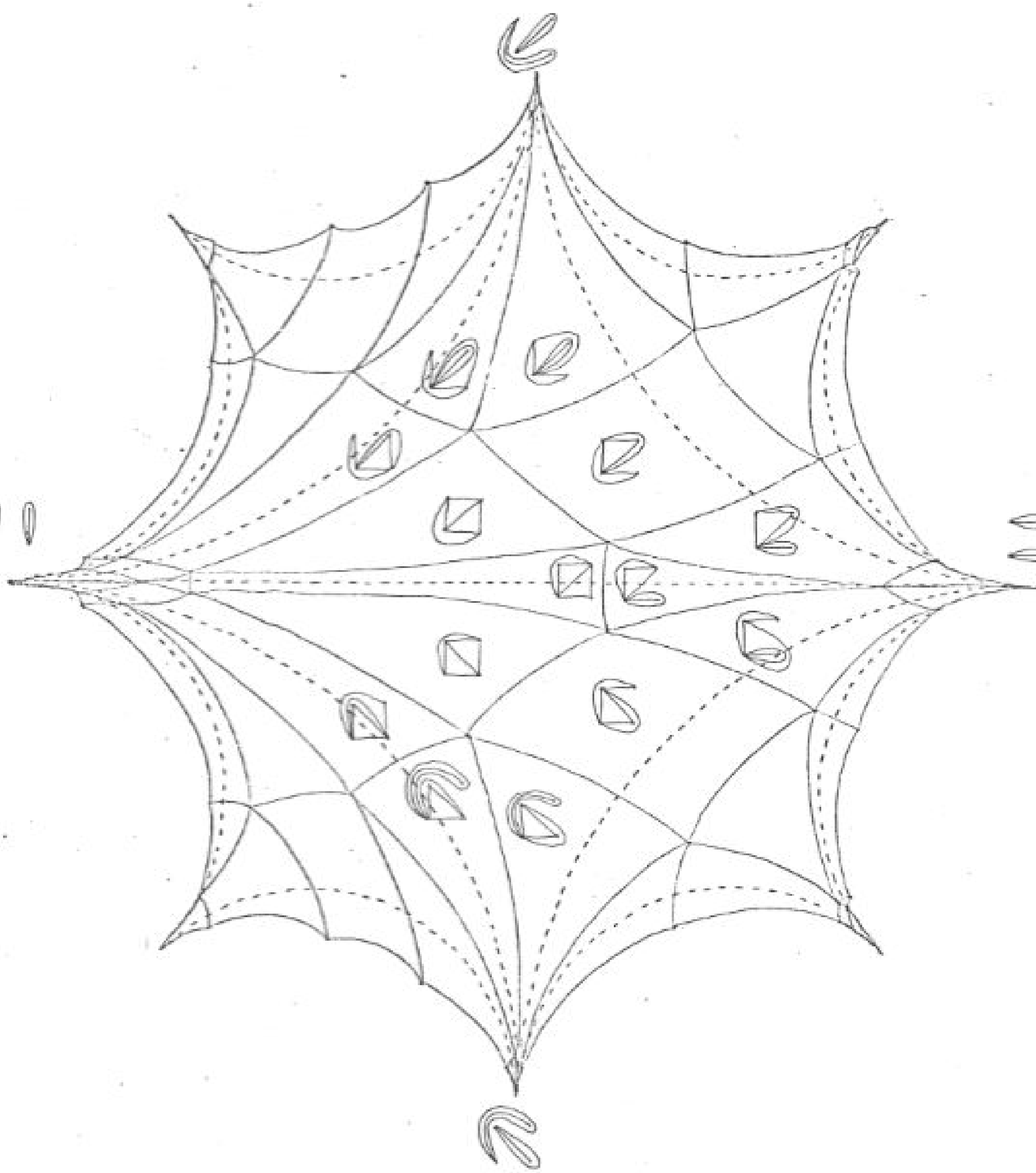
$$\begin{bmatrix} a+b+c & 0 \\ a+2b+c & a+b \end{bmatrix} - BCD$$



$$\begin{bmatrix} a+b+c+d & a \\ a+2b+c+d & a+b+c \end{bmatrix} - ABCD$$



$$\begin{bmatrix} a+b+c+d & a+b \\ 3a+b+2c+d & a+b+c \end{bmatrix} - AB'CD$$



$$\begin{bmatrix} a+b+d & a \\ 3a+2b+c+d & a+b \end{bmatrix} - A'B'CD$$