THE CLASSIFICATION PROBLEM FOR 3-MANIFOLDS

Program from ca. 1980:

1. Canonical decomposition into simpler pieces.
2. Explicit classification of special types of pieces.
3. Generic pieces are hyperbolic manifolds.

Will focus on the more topological aspects, 1 and 2.

Diff = PL = Top in dimension 3

Convention: manifolds are connected, orientable, and compact, possibly with boundary. "Closed" = "compact, no boundary".

For M a closed orientable 3-manifold, $\pi_1(M)$ determines all the homology groups $H_i(M)$:

$H_1(M)$ = abelianization of $\pi_1(M)$

$H_2(M) = H^1(M) = H_1(M)/\text{torsion}$

$H_3(M) = \mathbb{Z}$

$H_i(M) = 0$ for $i > 3$

Question: Does $\pi_1(M)$ determine M up to homeomorphism?

Poincaré Conjecture: $\pi_1(M) = 0 \Rightarrow M = S^3$.

Weaker question: Does $\pi_1(M)$ determine M up to homotopy equivalence?

$\pi_1(M) = 0 \Rightarrow M \simeq S^3$:

$H_1(M) = 0$

$\pi_2(M) = H_2(M) = 0$

$\pi_3(M) = H_3(M) = \mathbb{Z}$

gen by $f : S^3 \to M$, iso on $\pi_3 = H_3$

$\Rightarrow$ iso on H_n for all n

$\Rightarrow$ homotopy equivalence (Whitehead's Thm)

Prime Decomposition

Connected sum $P \# Q$: Delete interiors of closed balls in P and Q, then identify the two resulting boundary spheres.

Two essentially different ways to identify, depending on orientations. For orient<u>ed</u> manifolds $\#$ is unique.

$P \# Q = Q \# P$. Also associative.

$M \# S^3 = M$.

M is <u>prime</u> if $M = P \# Q \Rightarrow P = S^3$ or $Q = S^3$.

<u>Alexander's Theorem (1924)</u>. S^3 is prime: every (smooth) S^2 in S^3 bounds a ball on each side.

So no counterexamples to the Poincaré conjecture in S^3, since $\pi_1(P \# Q) = \pi_1(P) * \pi_1(Q)$ by van Kampen.

<u>Kneser's Theorem (1930)</u>. M compact, oriented $\Rightarrow$ M has a decomposition into primes, $M = P_1 \# \cdots \# P_n$, and this is unique up to insertion or deletion of S^3 summands.

Consequence: Choose prime manifolds P and Q neither of which has an orientation-reversing self-diffeomorphism. (There are lots of these.) Then the two ways of forming P # Q produce nonhomeomorphic manifolds having isomorphic π_1's.

Revised question: Does $\pi_1(M)$ determine M up to homeomorphism if M is prime (and closed)?

Rough classification of prime closed orientable 3-manifolds according to the size of π_1:

Type I: $\pi_1(M)$ finite.

Universal cover $\widetilde{M}$ is closed, simply-connected, hence $\widetilde{M} \simeq S^3$.

Only known examples are spherical manifolds S^3/Γ for Γ a finite subgroup of SO(4) acting freely on S^3 by rotations.

These were explicitly classified in the 1930s, using the two-sheeted covering $SO(4) \to SO(3) \times SO(3)$ to describe the finite subgroups of SO(4) in terms of the finite subgroups of SO(3).

The spherical manifolds with $\pi_1(M) = \Gamma$ cyclic are the lens spaces $L_{p/q}$ for $0 < p/q < 1$.

$\pi_1(L_{p/q}) = \mathbb{Z}_q$ but:

$$L_{p/q} = L_{p'/q} \iff p' \equiv \pm p^{\pm 1} \bmod q$$

so there can be many different lens spaces with the same π_1, e.g. at least $(q-1)/4$ if q is prime.

Taking orientations into account:

$$L_{p/q} = L_{p'/q} \iff p' \equiv p^{\pm 1} \mod q$$

Hence many lens spaces have no orientation-reversing homeomorphism, e.g. $L_{1/3}$.

Spherical S^3/Γ with Γ noncyclic: Several infinite families. All are uniquely determined by π_1.

Famous example: Poincaré homology sphere $S^3/\Gamma = SO(3)/I$ for I the icosahedral group, of order 60, so $|\Gamma| = 120$.

Type II: $\pi_1(M)$ infinite cyclic.

Only one such manifold that is closed and orientable: $S^1 \times S^2$.

Special properties of $S^1 \times S^2$:

1. It is the only prime orientable manifold that is not irreducible. (A manifold M is irreducible if every embedded S^2 in M bounds a ball in M.)

Proof: If M is prime but not irreducible, it contains an S^2 that is nonseparating: there is an S^1 in M intersecting the S^2 in one point transversely. A neighborhood N of $S^1 \cup S^2$ is $S^2 \times I$ with a 1-handle connecting its two boundary spheres. So $N = (S^1 \times S^2) - B^3$, hence $M = S^1 \times S^2 \# P$ for some P. Then M prime implies $M = S^1 \times S^2$.

2. $S^1 \times S^2$ is the only prime closed orientable manifold with π_2 nonzero.

This is an immediate consequence of the Sphere Theorem: In an orientable 3-manifold M, if $\pi_2(M)$ is nonzero then there is an embedded sphere in M that represents a nontrivial element of $\pi_2(M)$.

<u>Type III</u>: $\pi_1(M)$ infinite but not cyclic.

Hence M is irreducible.

Fact: An irreducible M with $\pi_1(M)$ infinite is a $K(\pi,1)$, i.e., the universal cover $\widetilde{M}$ is contractible.

Proof: $\widetilde{M}$ is simply-connected and has trivial homology groups: $H_2(\widetilde{M}) = \pi_2(\widetilde{M}) = \pi_2(M)$, and this is 0 by the Sphere Theorem and irreducibility. Since $\widetilde{M}$ is a noncompact 3-manifold we have $H_n(\widetilde{M}) = 0$ for $n > 2$. Whitehead's theorem then implies that $\widetilde{M}$ is contractible.

Thus if M is a closed manifold of Type III, $\pi_1(M)$ determines the homotopy type of M.

Some consequences:

1. $\pi_1(M)$ is torsionfree. Proof: If not, the covering space of M corresponding to a nontrivial finite cyclic subgroup $\mathbb{Z}_n \subset \pi_1(M)$ would be a finite-dimensional $K(\mathbb{Z}_n,1)$ CW complex, which cannot exist since infinite-dimensional lens spaces are $K(\mathbb{Z}_n,1)$'s and these have nontrivial homology in infinitely many dimensions.

2. If $\pi_1(M)$ is a free abelian group $\mathbb{Z}^n$ and M is a closed $K(\pi,1)$ then $n = 3$. (Example: the 3-torus T^3.)
Proof: The n-torus T^n is a $K(\mathbb{Z}^n,1)$ with $H_n(T^n) = \mathbb{Z}$ and $H_i(T^n) = 0$ for $i > n$, hence $\pi_1(M) = \mathbb{Z}^n \Rightarrow M \simeq T^n \Rightarrow n = 3$.

In particular, $\mathbb{Z}$ cannot be the fundamental group of a closed manifold of Type III, which shows that $S^1 \times S^2$ is the only prime closed orientable 3-manifold with π_1 infinite cyclic.

Borel Conjecture: A closed n-manifold $K(\pi,1)$ is determined up to homeomorphism by its fundamental group.

No counterexamples known in any dimension.

Waldhausen proved the conjecture for Haken 3-manifolds: irreducible M containing an embedded compact orientable surface S (not S^2) with $\pi_1(S) \to \pi_1(M)$ injective. (If ∂S is nonempty, assume S is properly embedded: $S \cap \partial M = \partial S$.) Such a surface S is called incompressible.

A Haken manifold can be split successively along a finite sequence of incompressible surfaces until all that remains are balls. Then one can do proofs by induction through the splittings. This works for Waldhausen's theorem (in a suitable relative form).

Some Haken manifolds:

1. Products $F \times S^1$ with F a compact orientable surface, not S^2.

2. More generally, fiber bundles over S^1 with fiber a compact orientable surface other than S^2.

3. More generally still, irreducible M with $H_1(M)$ infinite. ($H_1(M)$ is infinite if M is not closed and some component of ∂M has genus > 0.)

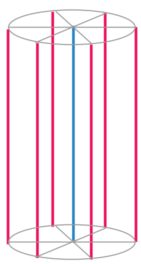
Are most Type III manifolds Haken? The known non-Haken manifolds of Type III all seem to be "small". Perhaps all "sufficiently large" irreducible 3-manifolds are Haken.

Conjecture: Every Type III manifold has a finite-sheeted cover that is Haken.

Seifert Manifolds

A special family of manifolds explicitly classified in the 1930s.

A Seifert fibering: Like an ordinary fiber bundle with circle fibers, but allow a finite number of exceptional circle fibers where the local model is:



Start with $D^2 \times I$ with the product fibering by intervals. Then glue the two ends together via a $2\pi p/q$ rotation of the disk. The central interval fiber becomes a circle fiber, but the other circle fibers are formed from q interval fibers.

Compact orientable manifolds that have a Seifert fibering are Seifert manifolds. These can have nonempty boundary consisting of tori, with product fiberings by circles.

Identifying each circle fiber to a point gives the base space, a compact surface which can be nonorientable.

Seifert fiber structures on a compact oriented manifold are classified by:

1. The topological type of the base surface.
2. The twists p/q (mod 1) at the exceptional fibers.
3. A rational "Euler number", in the case of Seifert fiberings of a closed manifold. This is the obstruction to a section.

In most cases the Seifert fibering of a Seifert manifold is unique up to isotopy. The exceptional cases can be listed explicitly, so one obtains an explicit classification of all Seifert manifolds.

It's easy to read off which ones have orientation-reversing homeomorphisms. (These just change the signs of the p/q's and the Euler number.)

A happy accident: All spherical 3-manifolds are Seifert manifolds, with base S^2 and at most 3 exceptional fibers.

The Type II manifold $S^1 \times S^2$ is also clearly a Seifert manifold.

The remaining Seifert manifolds are all of Type III, with the sole exception of $\mathbb{R}P^3 \# \mathbb{R}P^3$ which is not prime. This has $S^1 \times S^2$ as a 2-sheeted cover, and is the only nonprime manifold covered by a prime manifold.

Torus Decomposition

M irreducible $\iff$ M can't be simplified by splitting along spheres.

Try splitting along tori. Which tori?
"Trivial" tori: bound a solid torus $S^1 \times D^2$ or lie in a ball.
Splitting along these doesn't simplify the manifold.

These trivial tori are obviously compressible.

Fact: In an irreducible M these are the only compressible tori.
Follows from the Loop Theorem: If $\pi_1(\partial M) \to \pi_1(M)$ is not injective then there is a properly embedded disk $D \subset M$ with ∂D representing a nontrivial element of the kernel of $\pi_1(\partial M) \to \pi_1(M)$.

So try splitting along incompressible tori.

Torus Decomposition Theorem (Jaco-Shalen, Johannson):
If M is an irreducible compact orientable 3-manifold, then there is a finite collection of disjoint incompressible tori $T_1, \cdots, T_n$ in M such that splitting M along the union of these tori produces manifolds M_i that are either Seifert-fibered or atoroidal (every incompressible torus in M_i is isotopic to a torus component of ∂M_i). Furthermore, a minimal such collection of tori T_j is unique up to isotopy.

The collection of tori could be empty. This happens if M is itself either Seifert-fibered or atoroidal.

If the collection of tori T_j is not empty, the manifolds M_i will have boundary tori coming from the T_j's. There is no canonical way to fill these in with solid tori, so just leave the boundary tori unfilled. Thus the classification problem for closed manifolds inevitably leads to classifying manifolds with boundary tori.

M determines the M_i's uniquely, but not conversely. There is an $SL_2(\mathbb{Z})$ of choices for how to glue back together on each T_j. To classify the possible M's obtainable from given M_i's, need to know which homeomorphisms of ∂M_i extend to homeomorphisms of M_i.

This is known for Seifert-fibered M_i's. Hence can classify graph manifolds, where all M_i's are Seifert manifolds. (Waldhausen)

Remaining BIG problem: Classify irreducible manifolds that are atoroidal and not Seifert-fibered.

There are lots of these. For example, knot complements in S^3 are irreducible, and of the first million knots perhaps 99 percent are atoroidal and not Seifert-fibered.

Also most surface bundles.

Can construct many more examples via Dehn surgery: Given a link $L \subset M$, the union of disjoint embedded circles $L_1, \cdots, L_n$ with disjoint solid torus neighborhoods $N(L_i)$, first delete the interior of each $N(L_i)$ and then glue $N(L_i)$ back in via a homeomorphism $\partial N(L_i) \to \partial(M - \text{int}(N(L_i)))$. The resulting manifold only depends on how a meridian disk of $N(L_i)$ is glued in since glueing in the remaining ball is canonical. The glueing of the meridian disk is specified by the image of its boundary circle, a curve in a torus, determined by a slope p_i/q_i in $\mathbb{Q} \cup \{1/0\}$. Get a family of manifolds $M_L(p_1/q_1, \cdots, p_n/q_n)$ parametrized by points in a rational n-torus $(\mathbb{Q} \cup \{1/0\})^n$.

For a given M, most links L have $M - L$ irreducible, atoroidal, non-Seifert. Then $M_L(p_1/q_1, \cdots, p_n/q_n)$ is usually irreducible, atoroidal, and non-Seifert as well: namely for all n-tuples $(p_1/q_1, \cdots, p_n/q_n)$ in an open dense set in the rational n-torus.

Geometric examples: Hyperbolic manifolds $\mathbb{H}^3/\Gamma$ of finite volume are irreducible, atoroidal, and not Seifert-fibered.

The converse is the Hyperbolization Conjecture.

For further reading:

A Hatcher, *Basic Topology of 3-Manifolds,* Notes available at

http://www.math.cornell.edu/~hatcher

J Hempel, *3-Manifolds* (1976)

W Thurston, *Three-Dimensional Geometry and Topology* (1997)