Bianchi Orbifolds of Small Discriminant

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Let $\mathcal{O}_D$ be the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ of discriminant $D<0$. Then $PGL_2(\mathcal{O}_D)$ is a discrete subgroup of the isometry group $PSL_2(\mathbb{C})$ ($=PGL_2(\mathbb{C})$) of hyperbolic 3-space $\mathcal{H}^3$. The quotient space $\mathcal{H}^3/PGL_2(\mathcal{O}_D)=X_D$ is topologically a noncompact 3-manifold whose cusps (ends) are of the form $S^2\times[0,\infty)$. The number of cusps of X_D is known to be h_D, the class number of $\mathcal{O}_D$. So X_D is a closed manifold $\widehat{X}_D$ with h_D points removed.

For small D, including the 31 discriminants in the range $D>-100$, R. Riley has done computer calculations of the Ford fundamental domain F_D for the action of $PGL_2(\mathcal{O}_D)$ on $\mathcal{H}^3$. (See [5] for an account of the techniques; for about half of these D's, Bianchi had calculated the fundamental domains — by hand, presumably — almost a century ago, in [2],[3].) Riley's computer output includes how the faces of F_D are identified by elements of $PGL_2(\mathcal{O}_D)$. So it becomes a pleasant exercise in geometric visualization to try to recognize the manifold $\widehat{X}_D$. The results of carrying out this exercise for $D>-100$ are listed in Table I. (P^3 denotes real projective 3-space, $\sharp$ is connected sum.)

D	-3	-4	-7	-8	-11	-15	-19	-20	-23	-24	-31	-35	-39
$\widehat{X}_D$	S^3	S^3	S^3	S^3	S^3	S^3	S^3	S^3	S^3	S^3	S^3	S^3	S^3

-40	-43	-47	-51	-52	-55	-56	-59	-67	-68	-71	-79
P^3	P^3	S^3	S^3	P^3	P^3	S^3	S^3	$P^3\sharp P^3$	S^3	S^3	P^3

-83	-84	-87	-88	-91	-95
P^3	$S^1\times S^2$	$S^1\times S^2$	$P^3\sharp P^3\sharp P^3$	$P^3\sharp P^3$	P^3

Table I

There are exactly 19 D's in this range for which $\widehat{X}_D$ is the 3-sphere. Since $\pi_1 X_D$ ($=\pi_1\widehat{X}_D$) is $PGL_2(\mathcal{O}_D)/torsion$, the question of when $\widehat{X}_D$ is S^3 is equivalent (assuming no $\widehat{X}_D$'s are counterexamples to the Poincaré Conjecture) to when $PGL_2(\mathcal{O}_D)$ is generated by torsion. In the 19 cases when $\widehat{X}_D=S^3$, we have determined in addition the orbifold structure on X_D, namely, the embedded graph in X_D consisting of images under the quotient map $\mathcal{H}^3\to X_D$ of axes of rotations of torsion elements of $PGL_2(\mathcal{O}_D)$, each edge of this graph being labelled by the order of the corresponding torsion element. These orbifold structures are shown in the figures on the next page. (Labels “2” on edges are omitted. The small circles denote the cusp spheres. The sphere $S^3=\widehat{X}_D$ is regarded as 3-space compactified by a point at infinity.)

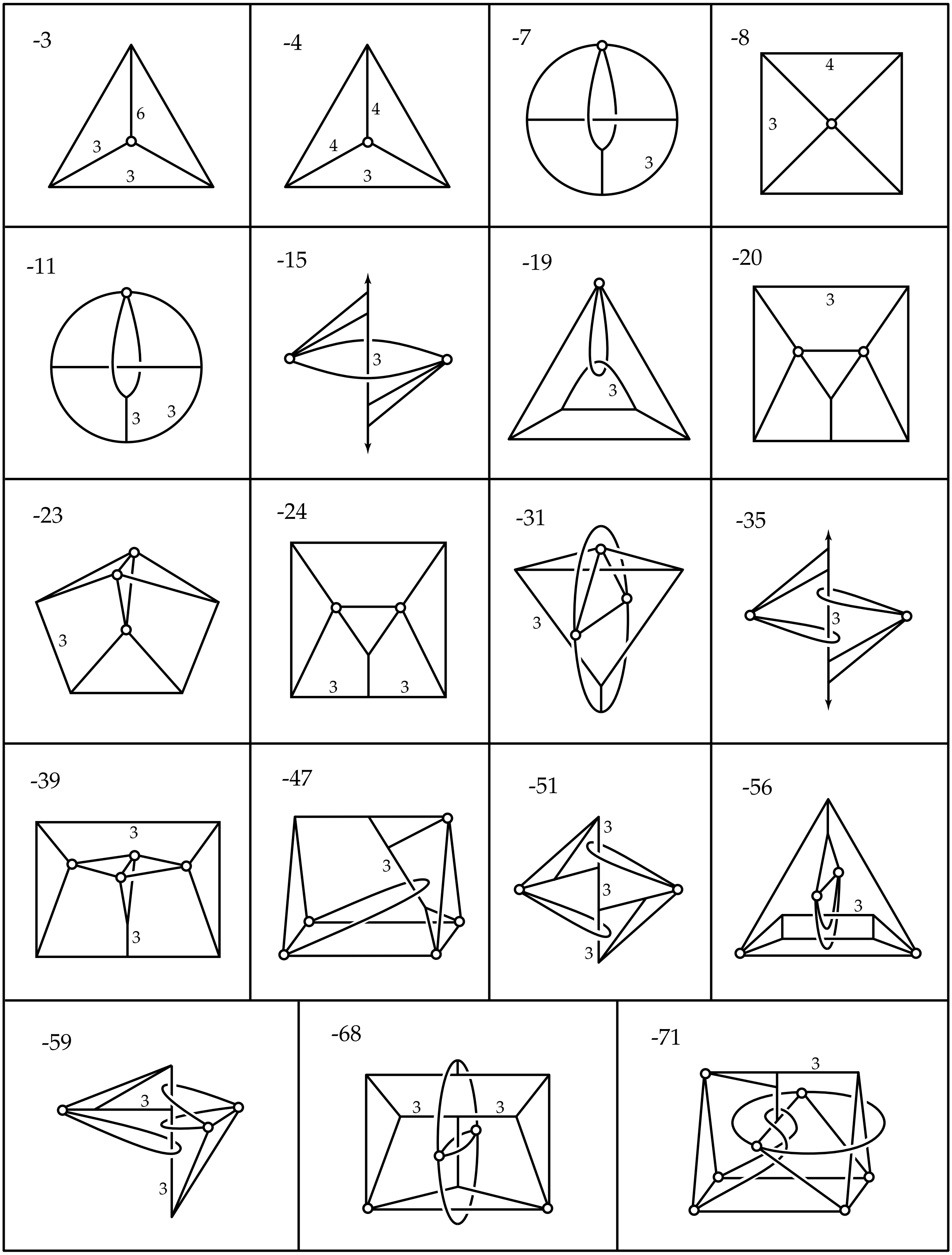
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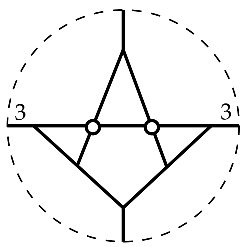
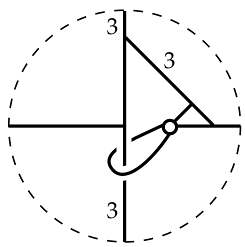
Table II below shows the orbifold structure on X_D in a few cases when $\widehat{X}_D \neq S^3$. In the top row are the first four cases when $\widehat{X}_D = P^3$. Here we view P^3 as a ball with antipodal points of its boundary sphere (indicated by the dashed-line circles) identified. The lower part of the Table represents the case $D = -84$, when $\widehat{X}_D = S^1 \times S^2$. The periodic extension of the graph shown, modulo its translation symmetries, gives the "singular" locus of the orbifold structure on X_{-84}, a graph lying on the torus $S^1 \times S^1$ which decomposes $S^1 \times S^2$ into two $S^1 \times D^2$'s. The vertical direction in the figure represents the meridian circles $\{x\} \times \partial D^2$ on these $S^1 \times D^2$'s.



-40



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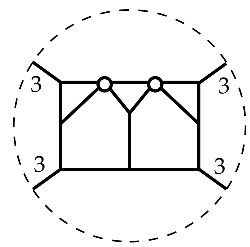

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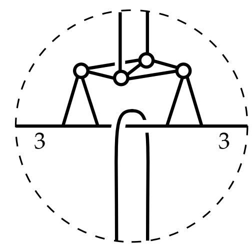
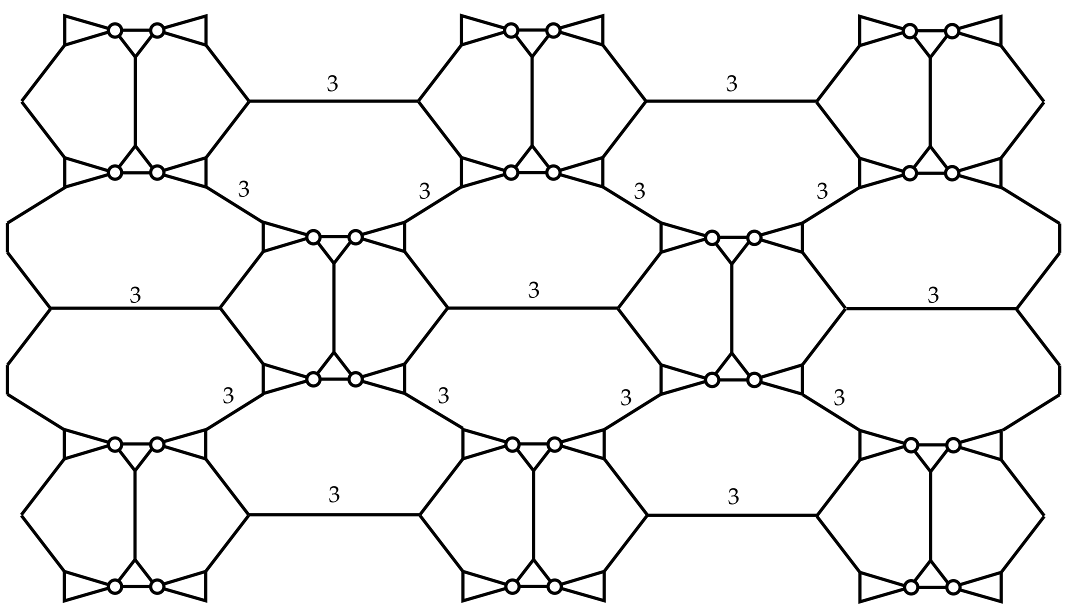


-55



-84

Table II

For the index two subgroup $PSL_2(\mathcal{O}_D)$ of $PGL_2(\mathcal{O}_D)$, the quotient space $Y_D = \mathcal{H}^3/PSL_2(\mathcal{O}_D)$ is a 2-sheeted branched cover of X_D, branched in such a way that the cusp spheres of X_D become cusp tori of Y_D (except for $D = -3, -4$, when they remain spheres). It turns out that in the 19 cases when $\widehat{X}_D = S^3$, this branching condition at cusps uniquely determines the branched covering $Y_D \to X_D$, and one can very easily by inspection determine the topological type of Y_D. This is given in Table III, in which Y_D is strictly speaking the interior of the compact manifold listed. (Notations: $B^3 =$ 3-ball, $D^2 =$ 2-disk, $T^2 =$ torus, $I = [0,1]$.)

D	Y_D
-3	B^3
-4	B^3
-7	$S^1 \times D^2$
-8	$S^1 \times D^2$
-11	$S^1 \times D^2$
-15	$S^1 \times D^2 \sharp S^1 \times D^2$
-19	$S^1 \times D^2$
-20	$S^1 \times D^2 \sharp S^1 \times D^2$
-23	$S^1 \times D^2 \sharp T^2 \times I$
-24	$S^1 \times D^2 \sharp S^1 \times D^2$

D	Y_D
-31	$S^1 \times D^2 \sharp T^2 \times I$
-35	$S^1 \times D^2 \sharp S^1 \times D^2 \sharp S^1 \times S^2$
-39	$S^1 \times D^2 \sharp S^1 \times D^2 \sharp T^2 \times I$
-47	$S^1 \times D^2 \sharp T^2 \times I \sharp T^2 \times I$
-51	$S^1 \times D^2 \sharp S^1 \times D^2 \sharp S^1 \times S^2$
-56	$S^1 \times D^2 \sharp S^1 \times D^2 \sharp T^2 \times I \sharp S^1 \times S^2$
-59	$S^1 \times D^2 \sharp T^2 \times I \sharp S^1 \times S^2$
-68	$S^1 \times D^2 \sharp S^1 \times D^2 \sharp T^2 \times I \sharp S^1 \times S^2$
-71	$S^1 \times D^2 \sharp T^2 \times I \sharp T^2 \times I \sharp T^2 \times I$

Table III

In the 14 cases that Y_D does not contain a connected summand $S^1 \times S^2$, the restriction map $H^1(Y_D;\mathbb{Z}) \to H^1(\partial Y_D;\mathbb{Z})$ is injective; in other words, "Y_D has no cuspidal cohomology." It is known (see [9],[4],[1],[6],[8]) that these are the only cases when this happens, for arbitrary $D < 0$.

Perhaps the first thing one notices about the pictures of the orbifolds X_D is the symmetries. In each case there is a reflectional symmetry through a plane parallel to the plane of the page. This reflection presumably corresponds to the $\mathbb{Z}_2$ extension of $PGL_2(\mathcal{O}_D)$ obtained by adjoining complex conjugation, the Galois automorphism of $\mathcal{O}_D$. When D has more than one distinct prime divisor, there is also a $180°$ rotational symmetry evident in the pictures. (This symmetry does not appear in the Ford domains, however.) Such a symmetry is predicted by general theory: Bianchi [3] already described a group $G_D \subset PGL_2(\mathbb{C})$ containing $PGL_2(\mathcal{O}_D)$ as a normal subgroup of finite index, with quotient $G_D/PGL_2(\mathcal{O}_D) \approx \mathcal{I}_D/2\mathcal{I}_D$, the mod 2 ideal class group (or genera group) of $\mathcal{O}_D$. According to Gauss, this quotient has $\mathbb{Z}_2$-rank equal to one less than the number of distinct prime divisors of D.

Even when $\widehat{X}_D$ is not S^3, $\widehat{X}_D$ may have S^3 as the quotient space corresponding to a finite extension of $PGL_2(\mathcal{O}_D)$, such as the group G_D above. This happens in the cases $D = -40, -52, -55$ in Table II, when $\widehat{X}_D = P^3$. It also happens for $D = -84$, when $\widehat{X}_D = S^1 \times S^2$ has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ ($\approx \mathcal{I}_{-84}/2\mathcal{I}_{-84}$) quotient which is S^3, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on X_{-84} restricting to the full symmetry group of the singular locus on the torus shown in Table II.

It appears that except for $D = -3, -4$, the remaining 17 orbifolds X_D with $\widehat{X}_D = S^3$ are Haken orbifolds [7]. That is, by repeatedly splitting open along incompressible 2-dimensional suborbifolds, X_D can be reduced to a disjoint union of finitely many orbifolds of the form $\mathbb{R}^3/\Gamma$ for Γ a finite subgroup of $SO(3)$ (acting on $\mathbb{R}^3$ as isometries). These splitting surfaces are all separating, so such a hierarchy for X_D yields a way of building up $PGL_2(\mathcal{O}_D)$ from finite subgroups of $SO(3)$ by iterated free product with amalgamation constructions. These hierarchies are in general far from unique. As a very simple example, the orbifold X_{-8} as drawn can be split successively along horizontal and vertical planes through the cusp, in either order, yielding the two structures

$$\big(O(24) *_{C(4)} D(8)\big) \ *_{C(3)*C(2)} \ \big(D(6) *_{C(2)} D(4)\big)$$

and

$$\big(O(24) *_{C(3)} D(6)\big) \ *_{C(4)*C(2)} \ \big(D(8) *_{C(2)} D(4)\big)$$

where $O(24)$ is the octahedral group, $D(2n)$ is the dihedral group of order $2n$, and $C(n)$ is the cyclic group of order n. In more subtle examples, not even the collection of finite subgroups of $SO(3)$ which start the iterated amalgamated free product construction is unique, though of course the noncyclic subgroups among these are unique, corresponding to the vertices in the singular locus of the orbifold structure.

In all cases except $D = -3, -4$, there is a splitting

$$PGL_2(\mathcal{O}_D) \approx PGL_2(\mathbb{Z}) *_A (?)$$

amalgamated over $A = PSL_2(\mathbb{Z})$, arising as follows. In the upper half-space model of $\mathcal{H}^3$, bounded below by the plane $\mathbb{C}$, there lies $\mathcal{H}^2$, the half-plane above $\mathbb{R}$. The orbifold $\mathcal{H}^2/PGL_2(\mathbb{Z})$ is a triangle with one vertex at the cusp at ∞. This triangle is embedded in X_D, and the boundary of a small regular neighborhood of this triangle is the surface corresponding to $PSL_2(\mathbb{Z})$ in the splitting above. This surface can be take to be totally geodesic in X_D. It should be of interest to find other totally geodesic incompressible surfaces in X_D, since these are more likely to be defined arithmetically. For example, as Riley has pointed out, the cuspidal classes in $H^1(Y_D;\mathbb{Z}) \approx H^2(Y_D, \partial Y_D;\mathbb{Z})$ found in [9],[4],[1] are represented by totally geodesic surfaces formed by the intersections of the Ford domain with certain planes parallel to $\mathcal{H}^2 \subset \mathcal{H}^3$. These non-separating surfaces in Y_D pass down to non-separating (totally geodesic) surfaces in X_D, which are often non-orientable. Since non-separating surfaces do not exist in S^3, it follows from [9],[4],[1] that the only values of $D < -100$ for which $\widehat{X}_D$ could be S^3 are $-119, -164, -191, -311, -356, -404, -479$, and -776. Riley's computer calculations eliminate -164 from this list.

References

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