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CONCORDANCE SPACES, HIGHER SIMPLE-HOMOTOPY THEORY, AND APPLICATIONS

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While much is now known, through surgery theory, about the classification problem for manifolds of dimension at least five, information about the automorphism groups of such manifolds is as yet rather sparse. In fact, it seems that there is not a single closed manifold M of dimension greater than three for which the homotopy type of the automorphism space Diff(M), PL(M), or TOP(M) in the smooth, PL, or topological category, respectively, is in any sense known. (As usual, Diff(M) is given the C^∞ topology, PL(M) is a simplicial group, and TOP(M) is the singular complex of the homeomorphism group with the compact-open topology.) Besides surgery theory, the principal tool in studying homotopy properties of these automorphism spaces is the *concordance space* functor $C(M) = \{$automorphisms of $M \times I$ fixed on $M \times 0\}$. This paper is a survey of some of the main results to date on concordance spaces.

Here is an outline of the contents. In §1 we describe how, in a certain stable dimension range, $C(M)$ is a homotopy functor of M, which we denote by $\mathscr{C}(M)$. The application to automorphism spaces is outlined in §2. In §3 we recall the explicit calculations which have been made for $\pi_0\mathscr{C}(M)$ and $\pi_1\mathscr{C}(M)$, along the lines pioneered by Cerf, and apply them in §4 to compute the group of isotopy classes of automorphisms of the n-torus, $n \geqq 5$. §5 is concerned with a stabilized version of $\mathscr{C}(M)$, defined roughly as $\Omega^\infty\mathscr{C}(S^\infty M)$, together with the curious equivalence of $\Omega^\infty\mathscr{C}_{\mathrm{PL}}(S^\infty M)$ with $\mathscr{C}_{\mathrm{PL}}(M)/\mathscr{C}_{\mathrm{Diff}}(M)$, due to Burghelea-Lashof (based on earlier fundamental work of Morlet). In §6, $\mathscr{C}_{\mathrm{PL}}(M)$ is "reduced" to higher simple-homotopy theory. This has some interest in its own right, e.g., it provides a fibered form of Wall's obstruction to finiteness. The important new work of Waldhausen relating $\mathscr{C}_{\mathrm{PL}}(M)$ to algebraic K-theory is outlined, very briefly and imperfectly, in §7. This seems to be the most promising area for future developments in the sub-

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ject. §8 describes how the expected extension of Waldhausen's work to $\mathscr{C}_{\text{Diff}}(M)$ leads to an apparent contradiction with the known calculation of $K_3(\mathbf{Z})$, using Igusa's work on $\pi_1\mathscr{C}_{\text{Diff}}(M)$. The only way out of this dilemma seems to be the rather unlikely prospect that Waldhausen's and Igusa's definitions of a certain K_3-type invariant, though both quite natural, do not agree.

Finally, two short appendices provide a product formula and iterated deloopings for concordance spaces.

1. The concordance functor $\mathscr{C}$. Let M be a compact manifold, and let $C(M)$ be the space of concordances of M, i.e., automorphisms of $M \times I$ fixed on $M \times 0$. According to the category, these will be diffeomorphisms, PL homeomorphisms, or topological homeomorphisms, and we write $C_{\text{Diff}}(M)$, $C_{\text{PL}}(M)$, or $C_{\text{TOP}}(M)$ when we wish to specify the category. It is known that when M is a PL manifold of dimension $\geqq 5$, the natural map $C_{\text{PL}}(M) \to C_{\text{TOP}}(M)$ is a homotopy equivalence **[3]**, **[18]** (essentially because $\text{Top}(n)/\text{PL}(n) \simeq \text{TOP}/\text{PL}$ for $n \geqq 5$). So we shall restrict our attention primarily to C_{Diff} and C_{PL}.

It is sometimes useful to replace $C(M)$ by the subspace $C(M \text{ rel } \partial M)$ consisting of concordances fixed on $\partial M \times I$. For example if $M \to N$ is a codimension-zero embedding, then there is induced $C(M \text{ rel } \partial M) \to C(N \text{ rel } \partial N)$ by extending concordances via the identity on $(N - M) \times I$. Of course, $C(M)$ can be identified with $C(M \text{ rel } \partial M)$ since $(M \times I, M \times 0)$ is isomorphic to $(M \times I, M \times 0 \cup \partial M \times I)$ by "bending around the corners". (This involves the usual smoothing of corners in Diff.) We will usually not distinguish between $C(M)$ and $C(M \text{ rel } \partial M)$, leaving the reader to determine by context which is meant.

Concordance spaces satisfy an important stability property:

THEOREM 1.1. *The inclusion* $C(M) \to C(M \times I), f \mapsto f \times \text{id}_I$, *is k-connected provided* $\dim M \gg k$.

This is proved in **[12]** for C_{PL}, in the range $\dim M \geqq 3k + 10$. With more care the same methods could probably be improved to yield $\dim M \geqq 2k + 8$. Burghelea-Lashof **[4]** reduced the theorem for C_{Diff} to the PL case, but with the dimension estimate doubled. Quite probably these stable dimension ranges can be considerably improved.

COROLLARY 1.2. $C(M) \to C(M \times I) \to C(M \times I^2) \to \cdots$ *is eventually an isomorphism on any* π_i.

DEFINITION. $\mathscr{C}(M) = \bigcup_n C(M \times I^n)$.

In the remainder of this section we shall show:

PROPOSITION 1.3. $\mathscr{C}(\text{—})$ *is a homotopy functor.*

The proof utilizes a transfer map for concordance spaces, which we now define. Let $p: E \to M$ be a locally trivial bundle in the category of compact (smooth or PL) manifolds. The transfer map will be $p^*: C(M) \to C(E)$. Observe first that a concordance $F \in C(M)$ determines (1) a function $f = (\text{proj}) \circ F: M \times (I, 0, 1) \to (I, 0, 1)$ and (2) a one-dimensional foliation $\mathscr{F} = F^{-1}$ (product foliation on $M \times I$) such that f restricts to a homeomorphism from each leaf of $\mathscr{F}$ to I. And conversely, a function f and a foliation $\mathscr{F}$ related in this way determine a concordance $F \in C(M)$.

So to define p^* we set $p^*(f) = f \circ (p \times id_I)$, and $p^*(\mathscr{F})$ we define via a local trivialization $M \times I \times$ (fiber of p) of $p \times \mathrm{id}_I$, setting $p^*(\mathscr{F}) = \mathscr{F} \times$ (point foliation). Any other local trivialization is related to this one by a transformation of the form $(m, t, x) \to (m, t, \phi_m(x))$, preserving $p^*(\mathscr{F})$, so $p^*(\mathscr{F})$ is well defined. This defines $p^*: C(M) \to C(E)$. It clearly commutes with the stabilization $M \to M \times I$, $E \to E \times I$, and so defines also $p^*: \mathscr{C}(M) \to \mathscr{C}(E)$.

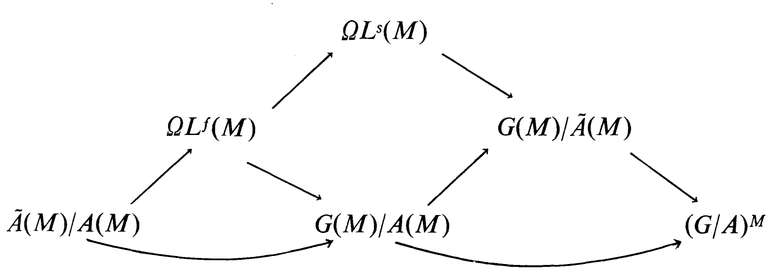
PROOF OF 1.3. Let $f: M \to N$ be a map. Replacing N by $N \times I^k$, k large, we may assume f is an embedding, with a neighborhood of its image in N a disc bundle $p: E \to M$. Then $f_*: \mathscr{C}(M) \to \mathscr{C}(N)$ is defined as the composition $i_* p^*$, where $i_*: \mathscr{C}(E) \to \mathscr{C}(N)$ is induced by the inclusion $i: E \to N$. With this definition of f_*, it is clear that $\mathscr{C}(—)$ becomes a homotopy functor on the category of compact manifolds (and continuous maps), or equivalently, on the category of finite complexes. One can trivially extend the domain of $\mathscr{C}$ to infinite (but locally finite) complexes by simply taking the direct limit over finite subcomplexes. (On a noncompact manifold this would amount to taking compactly supported concordances.)

2. Relation with automorphism groups. We will let $A(M)$ stand for one of the automorphism spaces Diff(M), PL(M), TOP(M) of diffeomorphisms, PL, or topological homeomorphisms of M. For convenience we assume M closed, though the results in this section hold also for compact M provided everything is taken rel ∂M.

The idea in trying to say something about the homotopy type of $A(M)$ is to compare it with $G(M)$, the H-space of self-homotopy equivalences of M, about which much more is currently known. For example, if M is a $K(\pi, 1)$, then as an easy application of obstruction theory, $G(M) \simeq \mathrm{Out}(\pi) \times K(\mathrm{Center}(\pi), 1)$, where Out (π) is the outer automorphism group of $\pi = \pi_1 M$, i.e., automorphisms modulo inner automorphisms.

One can interpolate between $A(M)$ and $G(M)$ the space $\tilde{A}(M)$ of block automorphisms of M. This is the simplicial group whose k-simplices are automorphisms of $M \times \Delta^k$ which leave invariant each $M \times$ (face of Δ^k). $\tilde{A}(M)$ contains $A(M)$ as the automorphisms of $M \times \Delta^k$ preserving projection to Δ^k. Similarly, one can define $\tilde{G}(M)$, but clearly $\tilde{G}(M) \simeq G(M)$, and we shall regard $\tilde{A}(M)$ as contained in $G(M)$.

According to surgery theory, there is a fibration (see §17A of **[29]**) $G(M)/\tilde{A}(M) \to (G/A)^M \to L^s(M)$ where $L^s(M)$ is Quinn's surgery space, G/A is G/O, G/PL, or G/TOP, and dim $M \geqq 5$. This fits into a braid of fibrations



where $\Omega L^f(M)$, the homotopy fiber of $G(M)/A(M) \to (G/A)^M$, can be regarded as a fibered-surgery form of $\Omega L^s(M)$. (See **[15]**. In this paper we will make no use of $\Omega L^f(M)$.)

So information about $G(M)/A(M)$ can be derived from surgery theory and information about $\tilde{A}(M)/A(M)$. The latter is intimately related to concordance spaces, as the following shows:

PROPOSITION 2.1. *There is a spectral sequence with* $E^1_{pq} = \pi_q C(M \times I^p)$ *converging to* $\pi_{p+q+1}(\tilde{A}(M)/A(M))$.

We outline the construction of this spectral sequence. An element of $\pi_k \tilde{A}(M)/A(M) = \pi_k(\tilde{A}(M), A(M))$ is represented by an automorphism of $M \times I^k$ which preserves projection to I^k over ∂I^k. Let $\bar{A}(M \times I^k)$ be the group of all such automorphisms, modulo the subgroup of those which preserve projection to I^k over all of I^k. Let $\bar{C}(M \times I^k)$ be the group of all automorphisms of $M \times I^k \times I$ which preserve projection to $I^k \times I$ over $I^k \times 0 \cup \partial I^k \times I$, modulo the subgroup of those preserving projection over all of $I^k \times I$. It is easy to verify that the natural map $C(M \times I^k) \to \bar{C}(M \times I^k)$ is a homotopy equivalence. There are fibrations $\bar{A}(M \times I^{k+1}) \to \bar{C}(M \times I^k) \to \bar{A}(M \times I^k)$ which give an exact couple

$$\begin{array}{ccc} \sum_{j,k} \pi_j \bar{A}(M \times I^k) & \xrightarrow{\partial} & \sum_{j,k} \pi_j \bar{A}(M \times I^{k+1}) \\ & \nwarrow \qquad \swarrow & \\ & \sum_{j,k} \pi_j \bar{C}(M \times I^k) & \end{array}$$

and hence a spectral sequence. The chain of homomorphisms

$$0 = \pi_k \bar{A}(M \times I^0) \xrightarrow{\partial} \pi_{k-1} \bar{A}(M \times I^1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \pi_0 \bar{A}(M \times I^k) \longrightarrow \pi_k(\tilde{A}(M), A(M))$$

gives a filtration of $\pi_k(\tilde{A}(M), A(M))$, according to how far back a given element can be pulled; successive obstructions to pulling back lie in $\pi_i \bar{C}(M \times I^{k-i-1})$, $i = 0, 1, \cdots, k-1$. The E^∞ term of the spectral sequence is associated to this filtration of $\pi_k(\tilde{A}(M), A(M))$.

The first differential is induced from $\delta: C(M \times I^p) \to C(M \times I^{p-1})$, where $\delta(f) = f \mid M \times I^p \times 1$, regarded as lying in $C(M \times I^{p-1})$. ($C(X)$ means $C(X \text{ rel } \partial X)$ here.) Now suppose $f = \Sigma g$, where $\Sigma: C(M \times I^{p-1}) \to C(M \times I^p)$ is stabilization. Note that in $C(X \text{ rel } \partial X)$, stabilization looks like:

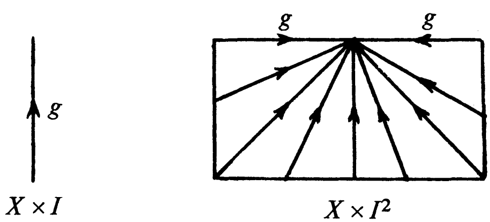


FIGURE 1

Thus $\delta(\Sigma g) = g + \bar{g}$, where the duality involution "—" on $C(X)$ is induced by reflecting I through its midpoint. We shall show in Appendix I that "—" anticommutes with Σ (up to homotopy), so we may define a Z_2-action on $\pi_* \mathscr{C}(M)$,

$[f] \to [\bar{f}]$, by letting it be $[f] \to (-1)^k[\bar{f}]$ on $\pi_* C(M \times I^k)$. (The sign of this involution on $\pi_* \mathscr{C}(M)$ depends on the parity of dim M.) These arguments yield:

PROPOSITION 2.2. *In the stable range* $p + \dim M \gg q$, $E^2_{pq} = H_p(\mathbf{Z}_2; \pi_q \mathscr{C}(M))$.

3. The brute force calculations of $\pi_0 \mathscr{C}(M)$ and $\pi_1 \mathscr{C}(M)$. The first major result about concordance spaces was Cerf's theorem that $\pi_0 C_{\mathrm{Diff}}(M) = 0$ if $\pi_1 M = 0$ and dim $M \geqq 5$ [6]. (In the PL category this is a much easier theorem, due to Rourke [24].) A refinement of Cerf's techniques yielded:

THEOREM 3.1 (Diff, PL, OR TOP). *For* dim $M \geqq 5$ *there is a natural exact sequence*

$$H_0(\pi_1 M; (\pi_2 M)[\pi_1 M]/(\pi_2 M)[1]) \longrightarrow \pi_0 C(M) \longrightarrow Wh_2(\pi_1 M) \oplus H_0(\pi_1 M; \mathbf{Z}_2[\pi_1 M]/\mathbf{Z}_2[1]) \longrightarrow 0.$$

If the first k-invariant of M (in $H^3(\pi_1 M; \pi_2 M)$) vanishes, this is a split short exact sequence (but the splitting is not natural).

For a π-module A, $H_0(\pi; A)$ is just A modulo the π-action. In the present case, $\pi_1 M$ acts in the usual way on $\pi_i M$ (hence by conjugation on itself) and trivially on $\mathbf{Z}_2$. $Wh_2(\pi_1 M)$ is a certain quotient of $K_2 \mathbf{Z}[\pi_1 M]$.

In the smooth category this theorem is proved in [14] for dim $M \geqq 6$. (The case dim $M = 5$ is due to K. Igusa.) However, when I wrote Part II of [14], I was not aware that I was using the vanishing of the k-invariant in Lemma 3.7, p. 262. (Igusa pointed out the error.) The lemma is actually false without the k-invariant hypothesis, but as far as I know the theorem may not require it. Volodin [27] announced the result without any restriction on k-invariants.

For the PL and TOP categories, one can show (see [23]) that $\pi_0 C(M)$ depends only on a neighborhood of the 3-skeleton of M, which can be smoothed, and then appeal to the equivalence $C_{\mathrm{PL}}(M) \simeq C_{\mathrm{TOP}}(M)$ for M PL, mentioned in §1, and to the results on $C_{\mathrm{PL}}(M)/C_{\mathrm{Diff}}(M)$ in §5 below. (Alternatively, the methods of [12] allow the Diff proof to be translated into PL.)

Igusa has gone much deeper with Cerf theory to obtain:

THEOREM 3.2 [16]. *Suppose the first two k-invariants of M vanish, and* dim M *is sufficiently large* ($\geqq 10$ *certainly suffices*). *Then there is an exact sequence*

$$0 \longrightarrow H_1(\pi_1 M; (\mathbf{Z}_2 \times \pi_2 M)[\pi_1 M]/(\mathbf{Z}_2 \times \pi_2 M)[1]) \oplus H_0(\pi_1 M; \Omega_2^{fr}(\Omega M)/\pi_3 M) \longrightarrow \pi_1 C_{\mathrm{Diff}}(M) \longrightarrow Wh_3(\pi_1 M) \longrightarrow 0.$$

Here $Wh_3(\pi)$ is defined as a certain quotient of $K_3 \mathbf{Z}[\pi]$. In particular, $Wh_3(0)$ is the cokernel of $\pi_3^S \to K_3 \mathbf{Z}$ which is $\mathbf{Z}_{24} \subsetneq \mathbf{Z}_{48}$ according to [20]. Thus $Wh_3(0) \approx \mathbf{Z}_2$.

COROLLARY 3.3. $\pi_1 C_{\mathrm{Diff}}(D^n)$ *has order* 4 (*n large*).

R. Lee has shown independently that $\pi_1 C_{\mathrm{Diff}}(D^n)$maps onto $Wh_3(0)$ for large enough n.

The mere fact that $\pi_1 C_{\mathrm{Diff}}(D^n)$ is nonzero is in many respects a striking result. It gives a new kind of difference between the smooth and PL categories, not traceable to exotic spheres. (Note that $C_{\mathrm{PL}}(D^n) \simeq *$ by the Alexander trick.) It follows that there are really two kinds of higher Whitehead groups, $Wh_i^{\mathrm{Diff}}(\pi)$ and $Wh_i^{\mathrm{PL}}(\pi)$, which coincide for $i \leqq 2$ only "by accident". For $i = 3$, $Wh_3^{\mathrm{PL}}(0) = 0$, but $Wh_3^{\mathrm{Diff}}(0)$ is the $Wh_3(0)$ above, which is $\boldsymbol{Z}_2$. More generally, it follows from 5.5 and 5.6 below that $\pi_1 C_{\mathrm{PL}}(M^n) \approx \pi_1 C_{\mathrm{Diff}}(M^n)/\pi_1 C_{\mathrm{Diff}}(D^n)$, n large, so one would expect that $Wh_3^{\mathrm{PL}}(\pi) \approx Wh_3^{\mathrm{Diff}}(\pi)/Wh_3^{\mathrm{Diff}}(0)$ for any π.

The calculations of $\pi_0 C(M)$ and $\pi_1 C(M)$ suggest that the n-type of $\mathscr{C}(M)$ depends only on the $(n+2)$-type of M. This is indeed true; see 5.2 below. Igusa **[17]** has shown this is best possible, in general: If in the Postnikov tower $\{M_k\}$ of M, the fibration $K(\pi_{n+2}M,\ n+2) \to M_{n+2} \to M_{n+1}$ has a homotopy-section, then $H_0(\pi_1 M; (\pi_{n+2}M)[\pi_1 M]/(\pi_{n+2}M)[1])$ is a direct summand of $\pi_n \mathscr{C}(M)$.

4. The isotopy classification of automorphisms of the n-torus. A good example for the preceding machinery is the calculation of $\pi_0 A(T^n)$, $n \geqq 5$, for $A = $ Diff, PL, or TOP. The steps go as follows.

(1) $G(T^n) \simeq T^n \times \mathrm{GL}(n, \boldsymbol{Z})$, and the map $A(T^n) \to G(T^n)$ has a section up to homotopy. Hence there is a split exact sequence

$$0 \longrightarrow \pi_1\, G(T^n)/A(T^n) \longrightarrow \pi_0\, A(T^n) \longrightarrow \mathrm{GL}(n, \boldsymbol{Z}) \longrightarrow 0.$$

(2) $G(T^n)/\widetilde{\mathrm{TOP}}(T^n) \simeq *$. For $n \geqq 5$ this is a result in surgery theory, see, e.g., **[25]**. However there is an elementary proof, using only the local contractibility of TOP(M), which works for all n **[19]**. Hence

$$\pi_1 G(T^n)/\mathrm{TOP}(T^n) \approx \pi_1 \widetilde{\mathrm{TOP}}(T^n)/\mathrm{TOP}(T^n).$$

(3) Using 2.2 (this is almost overkill), $\pi_1 \tilde{A}(T^n)/A(T^n) \approx \pi_0 C(T^n)/\{x \pm \bar{x}\}$, $n \geqq 5$. Then by 3.1, since $Wh_2(\pi_1 T^n) = 0$, we have

$$\pi_0 C(T^n) \approx \boldsymbol{Z}_2[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]/\boldsymbol{Z}_2[1]$$

where the t_i's generate $\pi_1 T^n$. According to **[14]**, the involution "—" on $\pi_0 C(T^n)$ is induced by $t_i \mapsto t_i^{-1}$. Hence $\pi_1\, \tilde{A}(T^n)/A(T^n) \approx \boldsymbol{Z}_2[t_1, \cdots, t_n]/\boldsymbol{Z}_2[1]$, $n \geqq 5$, independent of A.

(4) The preceding steps give the case $A = $ TOP. For $A = $ PL or Diff we consider the diagram

$$\begin{array}{ccccccc}
\longrightarrow & \pi_1\, \tilde{A}(T^n)/A(T^n) & \longrightarrow & \pi_1\, G(T^n)/A(T^n) & \longrightarrow & \pi_1\, G(T^n)/\tilde{A}(T^n) & \longrightarrow \\
 & \approx \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \pi_1 \widetilde{\mathrm{TOP}}(T^n)/\mathrm{TOP}(T^n) & \xrightarrow{\approx} & \pi_1\, G(T^n)/\mathrm{TOP}(T^n) & \longrightarrow & 0 &
\end{array}$$

This shows $\pi_1\, G(T^n)/A(T^n) \approx \pi_1\, \tilde{A}(T^n)/A(T^n) \oplus \pi_1\, G(T^n)/\tilde{A}(T^n)$.

(5) Again from surgery theory **[25]**,

$$\begin{aligned}
\pi_1\, G(T^n)/\mathrm{Diff}(T^n) &\approx hS(T^n \times I \text{ rel } \partial) \approx [\Sigma T^n, \mathrm{TOP}/O] \\
&\approx \sum_{i=0}^{n} H_{n-i}(T^n; \pi_{i+1}\, \mathrm{TOP}/O) \approx \sum_{i=0}^{n} \binom{n}{i} \Gamma_{i+1} \oplus \binom{n}{2} \boldsymbol{Z}_2.
\end{aligned}$$

Similarly,

$$\pi_1\, G(T^n)/\mathrm{PL}(T^n) \approx \binom{n}{2} \boldsymbol{Z}_2.$$

Thus we have:

THEOREM 4.1. *If $n \geqq 5$ there are split exact sequences*

$$0 \longrightarrow Z_2^\infty \longrightarrow \pi_0\,\mathrm{TOP}(T^n) \longrightarrow \mathrm{GL}(n, Z) \longrightarrow 0,$$

$$0 \longrightarrow Z_2^\infty \oplus \binom{n}{2} Z_2 \longrightarrow \pi_0\,\mathrm{PL}(T^n) \longrightarrow \mathrm{GL}(n, Z) \longrightarrow 0,$$

$$0 \longrightarrow Z_2^\infty \oplus \binom{n}{2} Z_2 \oplus \sum_{i=0}^{n} \binom{n}{i} \Gamma_{i+1} \longrightarrow \pi_0\,\mathrm{Diff}(T^n) \longrightarrow \mathrm{GL}(n, Z) \longrightarrow 0.$$

REMARKS. (1) This result was obtained also by Hsiang-Sharpe **[15]**.

(2) The same analysis allows one to compute $\pi_0\, A(T^n \times D^k \text{ rel } \partial)$, $n + k \geqq 5$, with no extra work. We leave this to the reader.

(3) The split extensions in 4.1 are nontrivial. The conjugation action of GL(n, Z) on

$$Z_2^\infty \approx Z_2[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]/Z_2[t_i + t_1^{-1}, \cdots, t_n + t_n^{-1}]$$

is induced by the usual action of GL(n, Z) on Z^n, the monomials. The action on $\binom{n}{2} Z_2 \approx H_{n-2}(T^n; Z_2)$ and $\Sigma\binom{n}{i}\Gamma_{i+1} \approx \Sigma H_{n-i}(T^n; \Gamma_{i+1})$ just comes from the action on T^n.

(4) The automorphisms in the subgroup $Z_2^\infty \subset \pi_0\, A(T^n)$ are diffeomorphisms which are concordant (smoothly) but not isotopic (even topologically) to the identity. These are rather delicate creatures. (a) They are annihilated by lifting to 2-fold covers (an observation of Laudenbach), hence by covers of any even order; also by certain odd order covers (depending on the diffeomorphism). (b) The product map $\pi_0\, A(T^n) \to \pi_0\, A(T^{n+1})$, $[f] \to [f \times \mathrm{id}_{S^1}]$, kills Z_2^∞ as we shall show in Appendix I. (c) On $T^n \# S^i \times S^{n-i}$ $(3 \leqq i \leqq n-3)$ any automorphism concordant to the identity is isotopic to the identity **[13]**. Thus $Z_2^\infty \subset \pi_0\, A(T^n)$ dies in $\pi_0\, A(T^n \# S^i \times S^{n-i})$.

(5) The subgroups $\Gamma_{i+1} \subset \pi_0\, A(T^n)$ are represented by diffeomorphisms of D^i rel ∂D^i cross the identity on a factor T^{n-i} of T^n. The elements of $\binom{n}{2} Z_2$ are represented by diffeomorphisms whose mapping tori are fake tori (homeomorphic but not PL homeomorphic to T^{n+1}).

To finish this section we will describe an explicit construction due to Farrell (unpublished), of a diffeomorphism f: $S^1 \times D^{n-1} \to S^1 \times D^{n-1}$ rel ∂ (n large) which is concordant to the identity but not obviously isotopic to the identity (everything rel ∂ here). To show that f is in fact not isotopic to the identity seems to require most of the machinery of **[14]**. A simpler proof of this would be quite welcome.

Embedding $S^1 \times D^{n-1}$ in T^n to represent an element $\alpha \in \pi_1 T^n$, one extends f on $S^1 \times D^{n-1}$ to a diffeomorphism f_α: $T^n \to T^n$ via the identity outside $S^1 \times D^{n-1}$. We leave it as an exercise to check that, in the subgroup $Z_2^\infty \approx Z_2[t_1, \cdots, t_n]/Z_2[1]$ of $\pi_0\, A(T^n)$, f_α represents the monomial generator $\alpha = t_1^{p_1} \cdots t_n^{p_n}$.

To construct f we will perform two embedded surgeries on the interior of the codimension one slice $D_0^{n-1} = * \times D^{n-1}$, producing a new disc $D_1^{n-1} \subset S^1 \times D^{n-1}$ with $\partial D_1^{n-1} = \partial D_0^{n-1}$ and $S^1 \times D^{n-1} - D_1^{n-1}$ still an n-ball, so that $D_1^{n-1} = f(D_0^{n-1})$ for some homeomorphism (in fact, diffeomorphism) f of $S^1 \times D^{n-1}$ rel ∂.

In a neighborhood of D_0^{n-1} label the two sides of D_0^{n-1} as $+$ and $-$. In the $+$ side, attach an embedded i-handle $D^i \times D^{n-i}$ to D_0^{n-1} in the trivial way. This effects a surgery on D_0^{n-1} to $\chi(D_0^{n-1})$, say. We could undo the effect of this surgery by now attaching an embedded $(i+1)$-handle $D^{i+1} \times D^{n-i-1}$ on the $+$ side of $\chi(D_0^{n-1})$, in the trivial way so that the surgered $\chi(D_0^{n-1})$ would be an $(n-1)$-disc isotopic to D_0^{n-1}. (All of this would occur near D_0^{n-1}.) The Farrell construction is to take instead a new embedding of the $(i+1)$-handle, but attached to $\chi(D_0^{n-1})$ in the same way so that $\chi(D_0^{n-1})$ is again surgered to a disc, this time the desired D_1^{n-1}. The new $(i+1)$-handle is obtained from the old by replacing the old core D^{i+1} by $D^{i+1} \# S^{i+1}$, the (interior) connected sum with a certain $S^{i+1} \subset (S^1 \times D^{n-1}) - \chi(D_0^{n-1})$. This S^{i+1} is constructed as follows. The core D^i of the i-handle can be completed to a sphere S^i by adding another i-disc on the $-$ side of D_0^{n-1}. In a neighborhood of this S^i, embed S^{i+1} so as to represent the Hopf map $S^{i+1} \to S^i$. For this we must assume $i \geqq 2$ and n large enough to get S^{i+1} actually embedded. Finally, to form the connected sum of D^{i+1}, which is on the $+$ side of $\chi(D_0^{n-1})$, with S^{i+1}, which is on the $-$ side, we must connect D^{i+1} to S^{i+1} by an arc which circles around the S^1 factor of $S^1 \times D^{n-1}$.

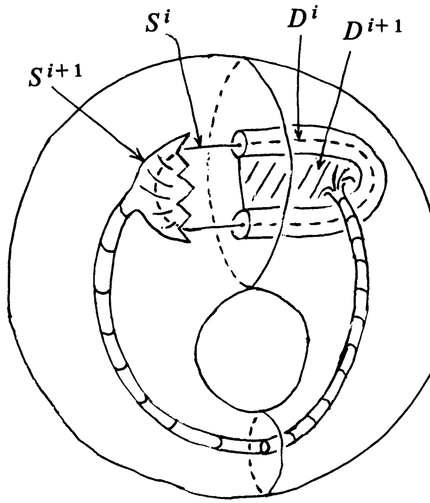


FIGURE 2

The construction actually gives a concordance from D_0^{n-1} to D_1^{n-1}, namely the trace of the two surgeries, that is,

$$(D_0^{n-1} \times [0, \tfrac{1}{3}]) \cup (h^i \times \tfrac{1}{3}) \cup (\chi(D_0^{n-1}) \times [\tfrac{1}{3}, \tfrac{2}{3}]) \cup (h^{i+1} \times \tfrac{2}{3}) \cup (D_1^{n-1} \times [\tfrac{2}{3}, 1])$$

in $S^1 \times D^{n-1} \times I$, where h^i and h^{i+1} are the i- and $(i+1)$-handles, respectively.

5. The functor $\mathscr{C}$ stabilized. Besides the stability Theorem 1.1, the other fundamental general result about concordance spaces is Morlet's Lemma of Disjunction. This is stated in terms of spaces of concordances of embeddings, defined as follows:

DEFINITION. Let $N \subset M$ be a proper submanifold ($N \cap \partial M = \partial N$). Then $CE(N, M)$ is the space of (proper) concordances $F: N \times I \hookrightarrow M \times I$ rel $N \times 0 \cup \partial N \times I$ with $F(N \times 1) \subset M \times 1$.

LEMMA OF DISJUNCTION (Diff OR PL). *Let $D^p \subset V^n$, $D^q \subset V^n$ be disjoint properly embedded discs, $n-p \geqq 3$, $n-q \geqq 3$. Then*

$$\pi_i(CE(D^p, V), CE(D^p, V - D^q)) = 0 \quad \textit{for } i \leqq 2n - p - q - 5.$$

For published proofs see **[5]** or **[22]**.

The lemma of disjunction can be reformulated in terms of *relative* concordance spaces $C(M, N) = C(M)/C(N)$, where $N \subset M$ is a codimension zero submanifold. (The relative $\mathscr{C}$ is defined similarly.) These give fibrations $C(N) \to C(M) \to C(M, N)$.

PROPOSITION 5.1 (EXCISION). *The lemma of disjunction is equivalent to*: $C(A, A \cap B) \to C(A \cup B, B)$ *is* $(k + l - 3)$*-connected if* $(A, A \cap B)$ *is* k*-connected* $(k > 1)$ *and* $(B, A \cap B)$ *is* l*-connected* $(l > 1)$.

PROOF. It suffices to consider the case that $A = M^n \cup (D^{k+1} \times D^{n-k-1})$ and $B = M^n \cup (D^{l+1} \times D^{n-l-1})$, i.e., disjoint $(k + 1)$- and $(l + 1)$-handles are attached to $M = A \cap B$. Then $C(A, A \cap B)$ can be regarded as the space $CE(D^{k+1} \times D^{n-k-1}, A)$ of concordances of $D^{k+1} \times D^{n-k-1}$ in A rel $\partial D^{k+1} \times D^{n-k-1}$. Similarly, $C(A \cup B, B) = CE(D^{k+1} \times D^{n-k-1}, A \cup B)$. Consider the fibrations

$$\begin{array}{ccccc}
CE(D^{k+1} \times D^{n-k-1}, A \text{ rel } 0 \times D^{n-k-1}) & \to & CE(D^{k+1} \times D^{n-k-1}, A) & \to & CE(0 \times D^{n-k-1}, A) \\
\downarrow & & \downarrow & & \downarrow \\
CE(D^{k+1} \times D^{n-k-1}, A \cup B \text{ rel } 0 \times D^{n-k-1} & \to & CE(D^{k+1} \times D^{n-k-1}, A \cup B) & \to & CE(0 \times D^{n-k-1}, A \cup B)
\end{array}$$

The two fibers are homotopy equivalent, essentially because one can shrink concordances of $D^{k+1} \times D^{n-k-1}$ rel $0 \times D^{n-k-1}$ to their germ near $0 \times D^{n-k-1}$. Choosing $D^p = 0 \times D^{n-k-1}$ and $D^q = 0 \times D^{n-l-1}$, the result now follows.

COROLLARY 5.2. *If* $X \to Y$ *is* k*-connected, the induced map* $\mathscr{C}(X) \to \mathscr{C}(Y)$ *is* $(k - 2)$*-connected.*

PROOF. We may take $Y = X \cup e^{k+1}$ with $k \geqq 2$. Splitting e^{k+1} down the middle, we get $e^{k+1} = e^k \cup e_1^{k+1} \cup e_2^{k+1}$; hence a fibration

$$\mathscr{C}(X \cup e^k, X) \longrightarrow \mathscr{C}(X \cup e^k \cup e_1^{k+1}, X) \longrightarrow \mathscr{C}(X \cup e^k \cup e_1^{k+1}, X \cup e^k)$$

with contractible total space. By induction on k, $\mathscr{C}(X \cup e^k, X)$ is $(k-3)$-connected, and therefore $\mathscr{C}(X \cup e^k \cup e_1^{k+1}, X \cup e^k)$ is $(k-2)$-connected. By excision (5.1)

$$\mathscr{C}(X \cup e^k \cup e_1^{k+1}, X \cup e^k) \longrightarrow \mathscr{C}(X \cup e^{k+1}, X \cup e^k \cup e_2^{k+1}) \simeq \mathscr{C}(X \cup e^{k+1}, X)$$

is $(2k - 3)$-connected, and so $\mathscr{C}(X \cup e^{k+1}, X)$ is $(k - 2)$-connected, as desired. (This argument is lifted from **[17]**.)

A purely formal consequence of 5.1, obtained by choosing A and B to be cones on $X = A \cap B$, is:

COROLLARY 5.3. *The natural suspension map* $\mathscr{C}(X, *) \to \Omega\, \mathscr{C}(SX, *)$ *is* $(2n - 2)$*-connected if* X *is* n*-connected.*

This allows us to make the following:

DEFINITION. $\mathscr{C}^S(X, *) = \lim_n \Omega^n \mathscr{C}(S^n X, *)$.

LEMMA 5.4. $\mathscr{C}^S_{\mathrm{Diff}}(X, *)$ *is contractible for all* X.

PROOF. $\pi_*\mathscr{C}^S(X, *)$ is a homology theory, since by 5.1 it satisfies the excision axiom. So it suffices to prove 5.4 when $X = S^n$, $* = D^n$. We claim: $C_{\text{Diff}}(S^n \times D^k, D^n \times D^k)$ is $(2n - 4)$-connected, for any k. For, an application of 5.1 gives

$$\pi_i C(S^n \times D^k, D^n \times D^k) \approx \pi_{i+1} C(S^{n+1} \times D^{k-1}, D^{n+1} \times D^{k-1}), \qquad i \leqq 2n - 4.$$

Iterating, we eventually get to $\pi_{i+k} C(S^{n+k}, D^{n+k})$. But in the smooth category, $C_{\text{Diff}}(S^{n+k}, D^{n+k}) \simeq CE_{\text{Diff}}(D^0, S^{n+k})$; and the latter space is contractible, by an amusing elementary argument which we leave to the reader. (*Hint.* Cap off $S^{n+k} \times I$ with an $(n + k + 1)$-ball attached to $S^{n+k} \times 1$.)

The main result of this section is due to Burghelea-Lashof [**4**]:

THEOREM 5.5. $\mathscr{C}_{\text{Diff}}(X, *) \to \mathscr{C}_{\text{PL}}(X, *) \to \mathscr{C}^S_{\text{PL}}(X, *)$ *is a fibration, up to homotopy.*

This is quite an amazing result. In the older approach to smooth concordance spaces, begun by Cerf, one studies k-parameter families of C^∞ functions $M \times I \to I$, and the first problem one encounters is the local one of understanding the singularities of codimension $\leqq k$. For example, when $k \leqq 4$ one encounters Thom's seven "elementary" catastrophes (these are all actually used in Igusa's work on $\pi_1 C(M)$ mentioned in §3). The complexity of these singularities increases rapidly with k, and they have only been completely classified (by Arnold) for relatively small values of k. So as an approach to smooth concordance spaces, this seems hopeless in general. Fortunately, the theorem gives an alternative approach in terms of PL concordance spaces, which are considerably more tractable as we shall see in §§6 and 7 below.

Theorem 5.5 is proved by considering the diagram

$$\begin{array}{ccccc}
\mathscr{C}_{\text{Diff}}(X, *) & \longrightarrow & \mathscr{C}_{\text{PL}}(X, *) & \longrightarrow & \mathscr{C}_{\text{PL}}(X, *)/\mathscr{C}_{\text{Diff}}(X, *) \\
\downarrow & & \downarrow & & \downarrow \simeq \\
\mathscr{C}^S_{\text{Diff}}(X, *) & \longrightarrow & \mathscr{C}^S_{\text{PL}}(X, *) & \xrightarrow{\simeq} & \mathscr{C}^S_{\text{PL}}(X, *)/\mathscr{C}^S_{\text{Diff}}(X, *).
\end{array}$$

The horizontal arrow labelled a homotopy equivalence is such because $\mathscr{C}^S_{\text{Diff}}(X, *) \simeq *$ by 5.4. The other homotopy equivalence follows from the fact that $\pi_* \mathscr{C}_{\text{PL}}(X, *)/\mathscr{C}_{\text{Diff}}(X, *)$ is a homology theory (hence already stable), which comes from fibered smoothing theory—see [**4**] for details.

COROLLARY 5.6. *The homology theory* $\pi_*\mathscr{C}^S_{\text{PL}}(X, *)$ *has coefficients* $\pi_{*-1}\mathscr{C}_{\text{Diff}}(*)$.

This follows by choosing $X = S^0$, since $\mathscr{C}_{\text{PL}}(S^0, *) = \mathscr{C}_{\text{PL}}(*) = \lim_n C_{\text{PL}}(D^n)$ is contractible by the Alexander trick.

Recall from §3 that $\pi_0\mathscr{C}_{\text{Diff}}(*) = 0$, but $\pi_1\mathscr{C}_{\text{Diff}}(*)$ is a group of order four. Nothing is yet known about $\pi_i\mathscr{C}_{\text{Diff}}(*)$ for $i > 1$. A very interesting question is whether or not $\pi_* \mathscr{C}_{\text{Diff}}(*)$ is all torsion.[1]

REMARK. According to [**4**], $\pi_*\mathscr{C}^S_{\text{PL}}(X, *)$ can also be described as the homology theory associated to the spectrum

[1]See note added in proof, below.

$$\frac{\mathrm{Top}(n+1)}{O(n+1)} \Big/ \frac{\mathrm{Top}(n)}{O(n)}.$$

6. Higher simple-homotopy theory. According to the usual pattern, one takes a geometric problem, reduces it to a homotopy problem, then tries to attack the homotopy problem by the big algebraic topology machine. In this section we describe part of the reduction of $\mathscr{C}_{\mathrm{PL}}$ to homotopy theory, though the homotopy theory which arises is not of the usual sort: It is a higher simple-homotopy theory, generalizing J. H. C. Whitehead [30]. In the following section (§7) this higher simple-homotopy theory is then related to more usual constructions in homotopy theory. One can anticipate that within a few years the algebraic topologists will have done something with this homotopy theory to make the whole program worthwhile.

As a motivation for higher simple-homotopy theory, we pose the following:

Problem 6.1 (*fibered obstruction to finiteness*). Let $\pi: E \to B$ be a fibration, with B a finite polyhedron and with fibers homotopy equivalent to a finite polyhedron X. Is π fiber-homotopy equivalent to a fibration $\pi': E' \to B$ such that

(a) E' is a finite polyhedron and π' is PL?

(b) π' is the projection of a locally trivial bundle with fiber a compact PL manifold M and structure group $\mathrm{PL}(M)$?

(c) E' is a compact ANR and π' is a proper map?

It can be shown that (a), (b), (c) are equivalent (see [7], [8] for (c)); we will focus on (a).

By a polyhedral version of the path space construction, one could easily construct a PL fibration $\pi': E' \to B$ fiber-homotopy equivalent to the given π, with E' an infinite polyhedron. On the other hand, if π' is required only to be a quasi-fibration, then E' can be taken to be a finite polyhedron (and in fact, one can take all fibers to be PL homeomorphic to X). The problem is to have both E' finite and π' satisfying the covering homotopy property.

Problem 6.1 can be reformulated as a lifting problem. As is well known, π is classified by a map $B \to BG(X)$, where $G(X)$ is the H-space of self-homotopy equivalences of X. One can construct a universal space $B(X)$ for PL fibrations of finite polyhedra, as follows. $B(X)$ is the simplicial set whose k-simplices are the PL maps $\pi: E \to \Delta^k$ satisfying the covering homotopy property, with E a finite polyhedron and fibers $\simeq X$. There is a natural forgetful map $B(X) \to BG(X)$, and Problem 6.1 becomes the lifting problem:

(6.2)
$$\begin{array}{ccc} & & B(X) \\ & \nearrow & \downarrow \\ B & \longrightarrow & BG(X) \end{array}$$

$B(X)$ has an amusing heuristic interpretation, as "the space of all finite polyhedra of the homotopy type of X", or more precisely as the singular complex of this "space". For the fibers of a k-simplex $\pi: E \to \Delta^k$ in $B(X)$ form a "continuous" k-parameter family of finite polyhedra, the "continuity" being expressed in the covering homotopy property for π.

The space $B(X)$ can be related to Whitehead's simple-homotopy theory. Recall the basic notion of an elementary collapse, sometimes written $L_0 \searrow^e L_1$, where L_0 is L_1 with a ball attached along one of its faces. Collapsing (projecting) the ball to this face induces the map $L_0 \to L_1$. The equivalence relation on finite complexes generated by elementary collapses is, by definition, simple homotopy equivalence.

A nice generalization of elementary collapse is given in the following definition (in the polyhedral category, for convenience):

DEFINITION. A PL map of finite polyhedra $f: L_0 \to L_1$ is a *simple map* if $f^{-1}(x) \simeq x$ for all $x \in L_1$.

The definition is due to M. M. Cohen [9], who used the term "contractible mapping". (In a somewhat more general setting the terminology "*CE* map" or "cell-like map" is also used.) Cohen proved that a simple map is a simple homotopy equivalence.

Let S be the category of finite polyhedra, with morphisms the simple maps. (It is easy to verify that simple maps are closed under composition.) The classifying space BS is then defined; it is the simplicial set whose k-simplices are the chains $L_0 \to L_1 \to \cdots \to L_k$ in S. Note that $\pi_0 BS$ is just the set of simple homotopy types of finite complexes, since simple maps generate the relation of simple homotopy equivalence.

Let BS_X denote those components of BS containing polyhedra of the homotopy type of the given X.

THEOREM 6.3 [12]. $BS_X \simeq B(X)$.

The map $BS_X \to B(X)$ can be defined as follows. A k-simplex of BS_X is a chain $L_0 \xrightarrow{f_1} L_1 \to \cdots \xrightarrow{f_k} L_k$. One forms its iterated mapping cylinder $M(f_1, \cdots, f_k)$, which is defined inductively as the ordinary mapping cylinder of the composition $M(f_1, \cdots, f_{k-1}) \to L_{k-1} \xrightarrow{f_k} L_k$. ($M(f_1)$ is the usual mapping cylinder.) Then one proves that the natural projection $M(f_1, \cdots, f_k) \to \Delta^k$ is a fibration if and only if the maps f_i are simple maps. The map $BS_X \to B(X)$ sends $L_0 \xrightarrow{f_1} L_1 \to \cdots \xrightarrow{f_k} L_k$ to $M(f_1, \cdots, f_k) \to \Delta^k$.

In view of the lifting problem (6.2), one is interested in the homotopy-theoretic fiber of $B(X) \to BG(X)$. This will be described using the following:

FUNDAMENTAL DEFINITION. $S(X)$ is the category whose objects are finite polyhedra containing the given finite subpolyhedron X as a deformation retract, and whose morphisms are the simple maps restricting to the identity on X.

THEOREM 6.4 [12]. *$BS(X) \to B(X) \to BG(X)$ is a fibration, up to homotopy.*

Thus obstructions to solving Problem 6.1 come from $\pi_* BS(X)$. Whitehead's fundamental theorem can be reformulated (much along the lines of [10]) as the calculation $\pi_0 BS(X) \approx Wh_1(\pi_1 X)$, the algebraic Whitehead torsion group, a quotient of $K_1 Z[\pi_1 X]$. As an example, let $f: X \to X$ be a homotopy equivalence with nonzero torsion, inducing the identity on $\pi_1 X$. Let $T(f)$ be the mapping torus, and $\pi: E \to S^1$ the path space construction applied to the obvious projection $T(f) \to S^1$, so that π is a fibration with fibers $\simeq X$. Then the answer to 6.1 is negative; the torsion of f is the obstruction (see [7] for more details).

Now to relate $BS(X)$ with concordance spaces, let X be a compact PL manifold.

Then there is a natural map of $BC_{\mathrm{PL}}(X)$ to the homotopy-fiber of $B(X) \to BG(X)$, since $BC_{\mathrm{PL}}(X)$ classifies PL bundles with fiber $X \times I$, trivialized on the subfiber $X \times 0$ (so, projection $X \times I \to X \times 0$ induces a fiber-homotopy trivialization). Stabilizing, one obtains a map of $B\mathscr{C}_{\mathrm{PL}}(X)$ to the homotopy-fiber of $B(X) \to BG(X)$.

THEOREM 6.5 [12]. *The natural map of $B\mathscr{C}_{\mathrm{PL}}(X)$ to the homotopy-fiber $BS(X)$ of $B(X) \to BG(X)$ is a homotopy equivalence onto the identity component of $BS(X)$.*

The other components of $BS(X)$ correspond to nontrivial h-cobordisms on X. Indeed, 6.5 can be regarded as a parametrized h-cobordism theorem, in the PL category.

A natural question is, does $B\mathscr{C}_{\mathrm{Diff}}(X)$ also have a categorical description? One obvious candidate is the category $E(X)$ whose objects are the same as those of $S(X)$, but whose morphisms are the finite compositions of elementary collapses. Then is $B\mathscr{C}_{\mathrm{Diff}}(X)$ homotopy equivalent to the identity component of $BE(X)$?

7. Waldhausen's "Quillenization" of $\mathscr{C}_{\mathrm{PL}}$. Waldhausen's basic idea is to imitate the exact sequence defining $Wh_1(\pi)$,

$$0 \to K_1(\mathbf{Z}) \oplus H_1(\pi) \to K_1\mathbf{Z}[\pi] \to Wh_1(\pi) \to 0$$

by constructing a diagram of fibrations

$$\begin{array}{ccccc} h(X; K(*)) & \longrightarrow & K(X) & \longrightarrow & Wh(X) \\ \downarrow & & \downarrow & & \downarrow \\ h(B\pi; K(\mathbf{Z})) & \longrightarrow & K(\mathbf{Z}[\pi]) & \longrightarrow & Wh(\pi) \end{array} \qquad \pi = \pi_1 X \tag{7.1}$$

where:

(a) $Wh(X)$ is a delooping of $BS(X)$, hence a double delooping of $\mathscr{C}_{\mathrm{PL}}(X)$. (This differs from the notation of **[12]**, where "$Wh(X)$" was equal to $BS(X)$.)

(b) $K(X)$ is "the algebraic K-theory of the topological space X." Waldhausen defines this (he calls it $A(X)$, but we have already used the letter A for automorphism spaces) using a very nice generalization of the Quillen Q-construction, but it seems that a plus-construction is also possible, and we give this definition. Let $GL(X_n)$ be the H-space of homotopy equivalences $X_n \vee_{j=1}^k S^i_j \to X_n \vee_{j=1}^k S^i_j$ rel X_n, stabilized over k and i, where $\{X_n\}$ is the Postnikov tower of X. (One needs $i \gg n$ in order to get canonical retractions $X_n \vee_{j=1}^k S^i_j \to X_n$ by means of which the stabilization with respect to i is defined.) We would like now to let $GL(X) = \lim_n GL(X_n)$, though strictly speaking, inverse limits are not generally defined. Nonetheless, there is an H-space $GL(X)$ which is the inverse limit of the system $\{GL(X_n)\}$ in the same sense that a space is the inverse limit of its Postnikov tower. For all practical purposes one can just choose large finite values $i \gg n \gg 0$, and only let $k \to \infty$, which is no problem. As an important special case, $GL(*)$ is (exactly) the H-space of base pointed homotopy equivalences of $\vee_{j=1}^k S^i_j$, stabilized over k and i in the obvious way.

PROPOSITION 7.2. *$\pi_0\, GL(X) \approx GL(\mathbf{Z}\pi_1 X)$, and for $i > 0$,*

$$\pi_i GL(X) \approx M(\Omega_i^{fr}(\Omega X)) = \lim_n M_n(\Omega_i^{fr}(\Omega X)),$$

where M_n is the additive group of $n \times n$ matrices and $M_n \subset M_{n+1}$ by adjoining zeros.

DEFINITION. $K(X) = BGL(X)^+ \times K_0(\mathbf{Z}\pi_1 X)$, where "+" is with respect to the commutator subgroup of $\pi_1 BGL(X)$.

(c) $\pi_* h(—; K(*))$ is the homology theory associated to $K(*)$ (all the spaces in 7.1 are infinite loopspaces).

(d) $K(\mathbf{Z}\pi) = B\mathrm{GL}(\mathbf{Z}\pi)^+ \times K_0(\mathbf{Z}\pi)$.

(e) $\pi_* h(—; K(\mathbf{Z}))$ is the homology theory associated to $K(\mathbf{Z})$.

(f) $Wh(\pi)$ is by definition a delooping of the homotopy-fiber of a natural map $h(B\pi; K(\mathbf{Z})) \to K(\mathbf{Z}\pi)$. See **[28]**. Waldhausen defines the higher Whitehead group $Wh_i(\pi)$ as $\pi_i Wh(\pi)$. For $i \leqq 2$ this agrees with earlier definitions. For $i = 3$ it is the $Wh_3^{\mathrm{PL}}(\pi)$ of §3. According to the main result of **[28]**, $Wh(\pi)$ is contractible for a large class of groups, e.g., free abelian groups.

(g) The map $K(X) \to K(\mathbf{Z}\pi_1 X)$ is induced from $BGL(X) \to BGL(\mathbf{Z}\pi_1 X)$, the first stage in the Postnikov tower for $BGL(X)$.

REMARK. The map $\Sigma_\infty \to GL(\mathbf{Z})$ factors through $GL(*)$, as permutations of the spheres in $\bigvee_{j=1}^\infty S_j^i$. Hence $\pi_*^S \to K_*(\mathbf{Z})$ factors through $\pi_* K(*)$.

An immediate consequence of the definitions and 7.2 is:

COROLLARY 7.3. *$K(B\pi) \to K(\mathbf{Z}\pi)$ is a $\mathbf{Q}$-equivalence. In particular, $K(*) \to K(\mathbf{Z})$ is a $\mathbf{Q}$-equivalence; hence also $Wh(B\pi) \to Wh(\pi)$.*

Thus if π is in Waldhausen's class of groups for which $Wh(\pi) \simeq *$, $Wh(B\pi)$ is a torsion space, in both senses! Going back to §§2 and 4, we can conclude from 7.3 as a very special case:

COROLLARY 7.4. *$\pi_i \mathrm{TOP}(T^n) \to \pi_i G(T^n)$ is an isomorphism* mod *torsion, for $i \ll n$. (And likewise for* PL.)

One good potential application of Waldhausen's work depends on the following:

CONJECTURE. *For any simply-connected X, $H_i(BGL(X))$ is finitely generated (f.g.) for all i.*

More generally, one might hope this is true if $\pi_1 X$ is finite.

PROPOSITION 7.5. *If the conjecture is true, then*

(a) *$\pi_i \mathrm{PL}(M^n)$ and $\pi_i \mathrm{TOP}(M^n)$ are f.g. for $i \ll n$ and M a simply-connected* PL *manifold.*

(b) *$\pi_i \mathrm{Diff}(M^n)$ is f.g. for $i \ll n$ and $\pi_1 M = 0$.*

PROOF. $H_*(BGL(M)) = H_* BGL(M)^+$ f.g. $\Rightarrow \pi_* K(M)$ f.g. $\Rightarrow \pi_* Wh(M)$ f.g. $\Rightarrow \pi_i C_{\mathrm{PL}}(M)$ f.g., $i \ll n$ (by 6.2) $\Rightarrow \pi_i \widetilde{\mathrm{PL}}(M)/\mathrm{PL}(M)$ f.g. (by 2.1) $\Rightarrow \pi_i G(M)/\mathrm{PL}(M)$ f.g. (since $\pi_* G(M)/\widetilde{\mathrm{PL}}(M)$ f.g. by surgery theory) $\Rightarrow \pi_i \mathrm{PL}(M)$ f.g. (since $\pi_* G(M)$ f.g.). And similarly for TOP. To go from PL to Diff, one uses 5.5.

One approach to studying $K(X)$ might be to take the Postnikov tower $\{BGL(X)_n\}$ for $BGL(X)$ and apply the plus construction. Thus one would have fibrations

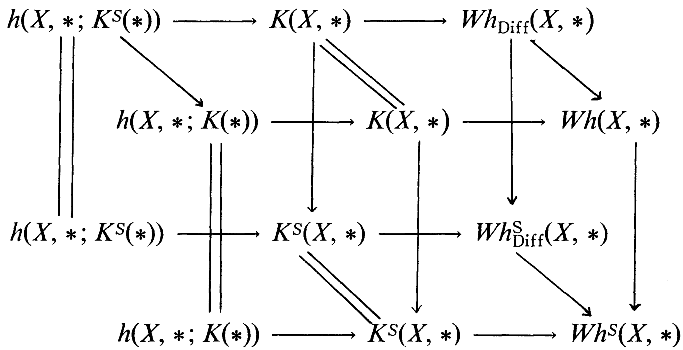
$$\begin{array}{ccccc}
K(M(\Omega_n^{fr}(\Omega X)), n+1) & \longrightarrow & BGL(X)_{n+1} & \longrightarrow & BGL(X)_n \\
\downarrow & & \downarrow & & \downarrow \\
F_{n+1}(X) & \longrightarrow & (BGL(X)_{n+1})^+ & \longrightarrow & BGL(X)_n)^+
\end{array}$$

and the question would be, what is the relation of the new fiber $F_{n+1}(X)$ to the old one $K(M(\Omega_n^{fr}(\Omega X)), n+1)$? It is not hard to see that $F_{n+1}(X)$ is n-connected, and that $\pi_{n+1}F_{n+1}(X)$ is just $M(\Omega_n^{fr}(\Omega X))$ modulo the conjugation action of $\mathrm{GL}(\boldsymbol{Z}\pi_1 X)$. This in turn is computable as $H_0(\pi_1 X; \Omega_n^{fr}(\Omega X))$, that is, $\Omega_n^{fr}(\Omega X)$ modulo the conjugation action of $\pi_1 X$ (see **[26]**). In view of 3.2, one might guess that $\pi_{n+2} F_{n+1}(X) \approx H_1(\pi_1 X; \Omega_n^{fr}(\Omega X))$. More generally, does $F_{n+1}(X)$ depend only on $M(\Omega_n^{fr}(\Omega X))$ and the conjugation action by $\mathrm{GL}(\boldsymbol{Z}\pi_1 X)$, at least in some stable range (below dimension $2n$, say)?

8. A hypothetical splitting. The following would seem to be a reasonable thing to hope for, and Waldhausen asserts in **[31]** that it is true:

Hypothesis. The map $K(X) \to Wh(X)$ of 7.1 factors through $Wh_{\mathrm{Diff}}(X)$, a double delooping of $\mathscr{C}_{\mathrm{Diff}}(X)$. (Recall that $Wh(X)$ is a double delooping of $\mathscr{C}_{\mathrm{PL}}(X)$.)

The hypothesis implies that there is a diagram of fibrations



The fibration containing $Wh^S(X, *)$ is obtained by stabilizing the one containing $Wh(X, *)$. The fiber $h(X, *; K(*))$ is the same for both since $\pi_* h(—; K(*))$ is a homology theory, and so is already stable. Similarly, π_* of the fiber of $K(X, *) \to Wh_{\mathrm{Diff}}(X, *)$ is a homology theory (since this is true for the PL analog $K(X, *) \to Wh(X, *)$ and $\pi_* \mathscr{C}_{\mathrm{PL}}(X, *)/\mathscr{C}_{\mathrm{Diff}}(X, *)$ is a homology theory, as mentioned after 5.5), so the fiber of $K(X, *) \to Wh_{\mathrm{Diff}}(X, *)$ is the same as the fiber of $K^S(X, *) \to Wh^S_{\mathrm{Diff}}(X, *) \simeq *$. Hence this fiber is $K^S(X, *)$, for which we are also using the notation $h(X, *; K^S(*))$.

The diagram yields:

COROLLARY OF THE HYPOTHESIS. $K(X, *) \simeq K^S(X, *) \times Wh_{\mathrm{Diff}}(X, *)$.

However, this seems to lead to a contradiction. Let $X = S^0$, and consider the diagram

$$\begin{array}{ccc} \pi_3 K(*) & \longrightarrow & \pi_1 \mathscr{C}_{\mathrm{Diff}}(*) \\ \downarrow & & \downarrow \\ K_3(\boldsymbol{Z}) & \longrightarrow & Wh_3^{\mathrm{Diff}}(0) \end{array}$$

where the maps to $Wh_3^{\mathrm{Diff}}(0)$ are those defined by Igusa (see §3). It seems reasonable to *suppose that the diagram commutes.* By the preceding corollary, Corollary 3.3, and **[20]**, we then get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_2 & \longrightarrow & \pi_1\mathscr{C}_{\mathrm{Diff}}(*) & \longrightarrow & Wh_3^{\mathrm{Diff}}(0) & \longrightarrow & 0 \\
 & & & & \downarrow & & \| & & \\
 & & & & K_3(Z) & \longrightarrow & Wh_3^{\mathrm{Diff}}(0) & \longrightarrow & 0 \\
 & & & & \wr\wr & & \wr\wr & & \\
 & & & & Z_{48} & & Z_2 & &
\end{array}$$

which is impossible. Can it be that the trouble is in the commutativity of the diagram?

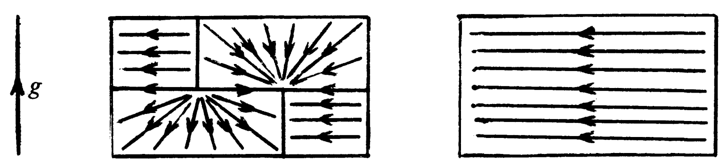
Appendix I. A product formula for concordances. Let M and N be compact manifolds. The product map $p: C(M) \to C(M \times N)$ is given by $p(F) = F \times id_N$. For $D^n \subset N^n$ there is also the inclusion-induced map $i: C(M) \xrightarrow{\Sigma^n} C(M \times D^n) \to C(M \times N)$.

PROPOSITION (Diff OR PL). *$p \simeq \chi(N) \cdot i$, where $\chi(N)$ is the Euler characteristic of N.*

For the proof it will be convenient to replace $C(X)$ by the space $C'(X)$ consisting of automorphisms of $(X \times I, X \times 0, X \times 1)$ which preserve projection to I over ∂X, modulo the subgroup of automorphisms preserving projection to I over all of X. It is easy to see that the natural map $C(X) \approx C(X \text{ rel } \partial X) \to C'(X)$ is a homotopy equivalence. In $C'(X)$ the duality involution "—" is easily defined as conjugation by $id_X \times r$, where $r: I \to I$ is reflection through the midpoint. The stabilization map $\Sigma: C'(X) \to C'(X \times I)$ looks just like it does in $C(X \text{ rel } \partial X)$; see Figure 1 in §2.

LEMMA. *Σ anticommutes with —, up to homotopy.*

A proof is suggested by the following pictures:



$X \times I$ (a) $X \times I^2$ (b) $X \times I^2$

FIGURE 3

The two large rectangles in (a) represent $\Sigma\bar{g}$ and $\overline{\Sigma g}$; the two smaller squares are isotopies (level preserving), hence trivial in $C'(X \times I)$. The whole of (a) is clearly isotopic to (b), itself an isotopy, hence trivial in $C'(X \times I)$. Thus $\Sigma\bar{g} + \overline{\Sigma g} \simeq 0$.

PROOF OF THE PROPOSITION. For convenience we choose the smooth category and assume N is closed. Let $g \in C'(M)$. As in Figure 4 below, first deform $g \times \mathrm{id}_N$ to G in $C'(M \times N)$ by shrinking vertically the support of g on each slice $M \times \{x\} \times$

$[\phi(x) - \varepsilon, \phi(x) + \varepsilon]$, where $\phi\colon N \to (0, 1)$ is a Morse function. Then deform G to G' by tilting the slices $M \times \{x\} \times [\phi(x) - \varepsilon, \phi(x) + \varepsilon]$ so that they are horizontal away from the critical points of ϕ, and so that near a critical point of index p, G' is just $(\overline{\Sigma^-})^p\,(\Sigma)^{n-p}\,(g)$. By the lemma, this equals $(-1)^p\, i\,(g)$. Summing over all critical points of ϕ gives the result.

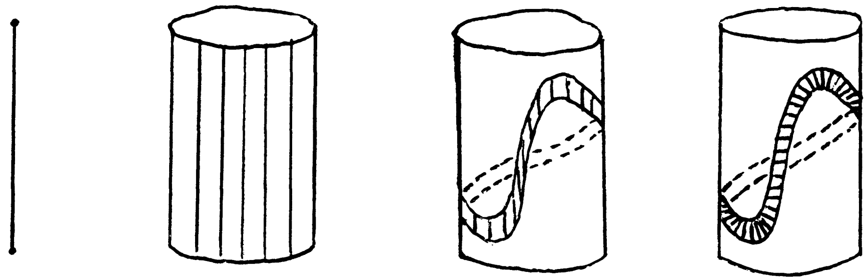


FIGURE 4

Appendix II. An infinite delooping of $\mathscr{C}(M)$. First, we mention the purely formal infinite delooping of $\mathscr{C}(M)$ coming from the obvious k-fold little cubes structure on $C(M \times I^k)$. This seems however not to have much geometric significance. More interesting is a delooping in terms of the spaces $C^b(M \times \mathbf{R}^k)$ of concordances $M \times \mathbf{R}^k \times I \to M \times \mathbf{R}^k \times I$ of bounded distance from the identity, given by the following:

PROPOSITION. *There are natural equivalences* $C^b(M \times \mathbf{R}^k \times I) \simeq \Omega C^b(M \times \mathbf{R}^{k+1})$ *compatible with stabilization and hence inducing* $\mathscr{C}^b(M \times \mathbf{R}^k) \simeq \Omega\mathscr{C}^b(M \times \mathbf{R}^{k+1})$. *In particular,* $\mathscr{C}(M) \simeq \Omega^k\mathscr{C}^b(M \times \mathbf{R}^k)$.

PROOF. Let $N = M \times \mathbf{R}^k$. A map $\lambda\colon C^b(N \times I) \to \Omega C^b(N \times \mathbf{R})$ can be defined as follows. Let $i\colon C^b(N \times I) \to C^b(N \times \mathbf{R})$ be induced by $I \subset \mathbf{R}$. Two null-homotopies of i are obtained by translating I to $+\infty$ and $-\infty$ in $\mathbf{R}$. This defines for each $f \in C^b(N \times I)$ the loop $\lambda(f)$ of concordances in $C^b(N \times \mathbf{R})$. To see that λ is a homotopy equivalence we consider the fibrations

$$\begin{array}{ccccc} C^b(N \times \mathbf{R} \operatorname{rel} N \times 0) & \longrightarrow & C^b(N \times \mathbf{R}) & \xrightarrow{\ r\ } & CE^b(N, N \times \mathbf{R}) \\ & & & & \Big\| \\ C^b(N \times [-\infty, 0] \operatorname{rel} \partial) & \longrightarrow & \mathscr{E} & \xrightarrow{\ r\ } & CE^b(N, N \times (-\infty, \infty)) \end{array}$$

where CE denotes concordances of embeddings,

$$\mathscr{E} = CE^b(N \times [-\infty,\ 0],\ N \times [-\infty, \infty] \operatorname{rel} N \times -\infty),$$

and the two maps r are restriction to $N \times 0$. (The boundedness condition assures the covering homotopy property for r.) Clearly $C^b(N \times \boldsymbol{R} \text{ rel } N \times 0)$ and $\mathscr{E}$ are contractible, and $C^b(N \times [-\infty, 0] \text{ rel } \partial) \simeq C^b(N \times I)$. It is not hard to check that the resulting equivalence $C^b(N \times I) \simeq \Omega C^b(N \times \boldsymbol{R})$ is given by λ.

THEOREM [2]. *The first k homotopy groups of $C^b(M^n \times \boldsymbol{R}^k)$, for $n + k \geqq 5$, are* $K_{-k+2}(\boldsymbol{Z}\pi_1 M), \cdots, K_{-1}(\boldsymbol{Z}\pi_1 M), \tilde{K}_0(\boldsymbol{Z}\pi_1 M), Wh_1(\pi_1 M)$.

Anderson and Hsiang have shown in [1] that the functors K_{-i} have an interesting geometric application to the problem of existence and uniqueness of triangulations of locally triangulable spaces. Roughly speaking, what they show is that, away from dimension 4 and apart from obstructions which arise already in the case of closed manifolds, the only other obstructions to the existence and uniqueness of triangulations are K_{-i} obstructions. It seems that this phenomenon should persist in the automorphism spaces of a polyhedron, namely that the differences between PL and TOP stem from the manifold case and from K_{-i} obstructions.

ADDED IN PROOF. Farrell and Hsiang, using Waldhausen's work, have now shown that $\pi_i \mathscr{C}_{\text{Diff}}(*) \otimes \boldsymbol{Q} \approx K_{i+2}(\boldsymbol{Z}) \otimes \boldsymbol{Q}$, which is now to be $\boldsymbol{Q}$ for $i = 3, 7, 11, \cdots$ and zero otherwise. From this they can compute the rank of $\pi_i \text{Diff}(S^n)$ and $\pi_i \text{Diff}(T^n)$, $i \ll n$. In particular, for odd $n \geqq 37$, neither $O(n+1) \to \text{Diff}(S^n)$ nor $\text{Diff}(T^n) \to G(T^n)$ is a $\boldsymbol{Q}$-equivalence!

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