# A 50-Year View of Diffeomorphism Groups

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**Question**: For a smooth compact manifold $M$ can one determine the homotopy type of its diffeomorphism group $\mathrm{Diff}(M)$?

Why this is interesting:

- Automorphisms are always interesting!
- $\mathrm{Diff}(M)$ is the structure group for smooth bundles with fiber $M$. Smooth bundles classified by maps to $\mathrm{BDiff}(M)$. Characteristic classes: $H^*(\mathrm{BDiff}(M))$.
- Relationship with algebraic K-theory.

Naive guess: $\mathrm{Diff}(M)$ has the homotopy type of a finite dimensional Lie group, perhaps the isometry group for some Riemannian metric on $M$.

Simplest case: $\mathrm{Diff}(S^n) \simeq O(n+1)$?

Remark: $\mathrm{Diff}(M)$ is a Fréchet manifold, locally homeomorphic to Hilbert space, hence it has the homotopy type of a CW complex and is determined up to homeomorphism by its homotopy type.

Outline of the talk:

I. Low dimensions ($\leq 4$)

II. High-dimensional stable range, e.g., $\pi_i \mathrm{Diff}(M^n)$ for $n >> i$. (Little known outside the stable range. Full homotopy type of $\mathrm{Diff}(M^n)$ not known for any compact $M^n$ with $n > 3$.)

III. Any dimension, but stabilize via $\#$. (Madsen-Weiss, ... )

## I. Low Dimensions.

Exercise: $\mathrm{Diff}(S^1) \simeq O(2)$ and $\mathrm{Diff}(D^1) \simeq O(1)$

**<u>Surfaces</u>**:

<u>Smale (1958)</u>:

$$\mathrm{Diff}(S^2) \simeq O(3) \qquad \mathrm{Diff}(D^2) \simeq O(2) \qquad \mathrm{Diff}(D^2 \,\mathrm{rel}\, \partial) \simeq *$$

These are equivalent via two general facts:

- $\mathrm{Diff}(S^n) \simeq O(n+1) \times \mathrm{Diff}(D^n \,\mathrm{rel}\, \partial)$
- Fibration $\quad \mathrm{Diff}(D^n \,\mathrm{rel}\, \partial) \to \mathrm{Diff}(D^n) \to \mathrm{Diff}(S^{n-1})$

Other compact orientable surfaces:

- $\mathrm{Diff}(S^1 \times S^1)$ has $\pi_0 = GL_2(\mathbb{Z})$, components $\simeq S^1 \times S^1$.
- $\mathrm{Diff}(S^1 \times I)$ has $\pi_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$, components $\simeq S^1$.
- Components of $\mathrm{Diff}(M^2)$ contractible in all other cases (Earle-Eells 1969, Gramain 1973).

$\pi_0 \mathrm{Diff}(M^2) =$ mapping class group, a subject unto itself. Won't discuss this.

<u>Problem</u>: Compute $H_* \mathrm{BDiff}(M^2)$, even with $\mathbb{Q}$ coefficients.

Non-orientable surfaces similar.

**<u>3-Manifolds</u>**:

<u>Cerf (1969)</u>: The inclusion $O(4) \hookrightarrow \mathrm{Diff}(S^3)$ induces an isomorphism on $\pi_0$. Equivalently, $\pi_0 \mathrm{Diff}(D^3 \,\mathrm{rel}\, \partial) = 0$.

Essential for smoothing theory in higher dimensions.

Extension of Cerf's theorem to higher homotopy groups (H 1983):

$$\mathrm{Diff}(S^3) \simeq O(4) \qquad \mathrm{Diff}(D^3) \simeq O(3) \qquad \mathrm{Diff}(D^3 \,\mathrm{rel}\, \partial) \simeq *$$

Another case (H 1981):

$$\mathrm{Diff}(S^1 \times S^2) \simeq O(2) \times O(3) \times \Omega SO(3)$$

In particular $\mathrm{Diff}(S^1 \times S^2)$ is not homotopy equivalent to a Lie group since $H_{2i}(\Omega SO(3)) \neq 0$ for all $i$.

Reasonable guess: $\mathrm{Diff}(M)$ for other compact orientable 3-manifolds that are prime with respect to connected sum should behave like for surfaces.

This is known to be true in almost all cases:

- $\mathrm{Diff}(M^3)$ has contractible components unless $M$ is Seifert fibered via an $S^1$ action.
  - Haken manifolds: H, Ivanov 1970s.
  - Hyperbolic manifolds: $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$. Gabai 2001.

- If $M$ is Seifert fibered via an $S^1$ action, the components of $\mathrm{Diff}(M)$ are usually homotopy equivalent to $S^1$. Most cases covered by Haken manifold result. Exceptions:
  - Components of $\mathrm{Diff}(S^1 \times S^1 \times S^1) \simeq S^1 \times S^1 \times S^1$
  - Components of $\mathrm{Diff}(S^1 \times S^1 \times I) \simeq S^1 \times S^1$.
  - Spherical manifolds. Expect $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$ from the case $M = S^3$. Known for lens spaces and dihedral manifolds: Ivanov in special cases, Hong-Kalliongis-McCullough-Rubinstein in general. Unknown for tetrahedral, octahedral, dodecahedral manifolds, including the Poincaré homology sphere.
  - Also unknown for some small nilgeometry manifolds.
  - Proved for the small non-Haken manifolds with two other geometries, $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL}_2(\mathbb{R})$, by McCullough-Soma (2010).

- $\pi_0\mathrm{Diff}(M)$ known for all prime $M$.

Non-prime 3-manifolds:

Say $M = P_1 \# \cdots \# P_k \# (\#_n S^1 \times S^2)$ with each $P_i \neq S^1 \times S^2$.

There is a fibration

$$CS(M) \to \mathrm{BDiff}(M) \to \mathrm{BDiff}(\coprod_i P_i)$$

where $CS(M)$ is a space parametrizing all the ways of constructing $M$ explicitly as a connected sum of the $P_i$'s and possibly some $S^3$ summands. Allow connected sum of a manifold with itself to get $S^1 \times S^2$ summands.

Idea due to César de Sá and Rourke (1979), carried out fully (with different definitions) by Hendriks and Laudenbach (1984).

$CS(M)$ is essentially a combinatorial object, $\simeq$ finite complex.

Easily get a finite generating set for $\pi_0 \mathrm{Diff}(M)$ from generators for each $\pi_0 \mathrm{Diff}(P_i)$ and generators for $\pi_1 CS(M)$.

$\pi_1 \mathrm{Diff}(M)$ usually not finitely generated (McCullough), from $\pi_2 CS(M)$ being not finitely generated.

More work needed to understand $CS(M)$ better.

**4-Manifolds**:

Situation seems similar to current state of the classification problem for smooth 4-manifolds, but $\mathrm{Diff}(M^4)$ has been studied much less.

Just as the smooth 4-dimensional Poincaré conjecture is unknown, so is it unknown whether $\mathrm{Diff}(D^4 \operatorname{rel} \partial)$ is contractible or even connected.

Quinn (1986): If two diffeomorphisms of a closed simply-connected smooth 4-manifold are homotopic, then after connected sum with a large enough number of copies of $S^2 \times S^2$ they become isotopic. (In the topological category this stabilization is not necessary.)

Ruberman (1998): Examples where the stabilization is necessary. More recent refinement by Auckly-Kim-Melvin-Ruberman (2014): For a connected sum of at least 26 copies of $\mathbb{C}P^2$ with certain orientations, there exist infinitely many isotopy classes of diffeomorphisms which all become isotopic after connected sum with a single copy of $S^2 \times S^2$.

## II. High Dimensional Stable Range.

<u>**Dimension** $\geq 5$</u>.

Gluing map $\pi_0 \mathrm{Diff}(D^n \operatorname{rel} \partial) \to \Theta_{n+1}$, group of exotic $(n+1)$-spheres.

Surjective for $n \geq 5$ by the h-cobordism theorem (Smale 1961).

Injective for $n \geq 5$ by Cerf (1970):

<u>**Theorem**</u>. Let $C(M) = \mathrm{Diff}(M \times I \operatorname{rel} M \times 0 \cup \partial M \times I)$. If $\pi_1 M^n = 0$ and $n \geq 5$ then $\pi_0 C(M) = 0$.

Elements of $C(M)$ are called concordances or pseudoisotopies.

Since $\Theta_{n+1} \neq 0$ for most $n$, it follows that $\pi_0 \mathrm{Diff}(D^n \operatorname{rel} \partial) \neq 0$ for most $n \geq 5$. Exceptions: $n = 5, 11, 60$. Others?

Cerf's theorem implies $\pi_1 \mathrm{Diff}(D^n \operatorname{rel} \partial) \to \pi_0 \mathrm{Diff}(D^{n+1} \operatorname{rel} \partial)$ surjective for $n \geq 5$. Thus $\mathrm{Diff}(D^n \operatorname{rel} \partial)$ also noncontractible for $n = 5, 11, 60$.

In fact $\mathrm{Diff}(D^n \operatorname{rel} \partial)$ is noncontractible for all $n \geq 5$. This was probably known 30 or 40 years ago, but a stronger statement is:

Crowley-Schick (2012): $\pi_i \mathrm{Diff}(D^n \operatorname{rel} \partial) \neq 0$ for infinitely many $i$, for each $n \geq 7$.

<u>Question</u>: Is $\pi_2 \mathrm{Diff}(D^4 \operatorname{rel} \partial) \to \pi_1 \mathrm{Diff}(D^5 \operatorname{rel} \partial)$ nontrivial?

Usually $\pi_0 C(M) \neq 0$ when $\pi_1 M \neq 0$ and $n \geq 5$ (H and Igusa, 1970s).

<u>Examples</u>:

- $\pi_0 \mathrm{Diff}(S^1 \times D^{n-1} \operatorname{rel} \partial) \supset \mathbb{Z}_2^\infty$ for $n \geq 5$.
- $\pi_0 \mathrm{Diff}(T^n) \supset \mathbb{Z}_2^\infty$ for $n \geq 5$.

These are diffeomorphisms that are homotopic to the identity ( $\operatorname{rel} \partial$ ) but not isotopic to the identity, even topologically.

<u>Concordance Stability (Igusa 1988)</u>: $C(M^n) \hookrightarrow C(M^n \times I)$ induces an isomorphism

on $\pi_i$ for $n >> i$.

Denote the limiting object by $\mathcal{C}(M) = \cup_k C(M \times I^k)$.

**The Big Machine**.

Main foundational work: Waldhausen in the 1970s and 80s, with many other subsequent contributors.

Idea: Compare $\text{Diff}(M)$ with a larger space $\widetilde{\text{Diff}}(M)$, the simplicial space whose $k$-simplices are diffeomorphisms $M \times \Delta^k \to M \times \Delta^k$ taking each $M \times \text{face}$ to itself but not necessarily preserving fibers of projection to $\Delta^k$.

$\widetilde{\text{Diff}}(M)$ is accessible via surgery theory.

Fibration

$$\text{Diff}(M) \to \widetilde{\text{Diff}}(M) \to \widetilde{\text{Diff}}(M)/\text{Diff}(M)$$

Weiss-Williams (1988): In the stable range,

$$\widetilde{\text{Diff}}(M)/\text{Diff}(M) \simeq B\mathcal{C}(M)//\mathbb{Z}_2 = (B\mathcal{C}(M) \times S^\infty)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts on $C(M)$ by switching the ends of $M \times I$ (and renormalizing).

Nice properties of $\mathcal{C}$:

- Definition extends to arbitrary complexes $X$.
- A homotopy functor of $X$.
- An infinite loopspace.

$\mathcal{C}(X)$ is related to algebraic K-theory via Waldhausen's 'algebraic K-theory of topological spaces' functor $A(X)$.

Special case with an easy definition: Let $G(\vee_k S^n)$ be the monoid of basepoint-preserving homotopy equivalences $\vee_k S^n \to \vee_k S^n$. Stabilize this by letting $k$ and $n$ go to infinity, producing a monoid $G(\vee_\infty S^\infty)$. Then $A(*) = BG(\vee_\infty S^\infty)^+$ where $+$ denotes the Quillen plus construction.

The homomorphism $G(\vee_\infty S^\infty) \to \pi_0 G(\vee_\infty S^\infty) = GL_\infty(\mathbb{Z}) = \cup_k GL_k(\mathbb{Z})$ induces a map $A(*) \to K(\mathbb{Z}) = BGL_\infty(\mathbb{Z})^+$.

More generally there is a natural map $A(X) \to K(\mathbb{Z}[\pi_1 X]) = BGL_\infty(\mathbb{Z}[\pi_1 X])^+$.

**Theorem (Waldhausen 1980s)**: $A(X) \simeq \Omega^\infty S^\infty(X_+) \times \mathrm{Wh}(X)$ where $\mathcal{C}(X) \simeq \Omega^2 \mathrm{Wh}(X)$ and $X_+ = X \cup \text{point}$.

Dundas (1997): There is a homotopy-cartesian square relating the map $A(X) \to K(\mathbb{Z}[\pi_1 X])$ to topological cyclic homology $TC(-)$:

$$\begin{array}{ccc} A(X) & \to & K(\mathbb{Z}[\pi_1 X]) \\ \downarrow & & \downarrow \\ TC(X) & \to & TC(\mathbb{Z}[\pi_1 X]) \end{array}$$

This means the homotopy fibers of the two horizontal maps are the same.

Thus the difference between $A(X)$ and $K(\mathbb{Z}[\pi_1 X])$ can be measured in terms of topological cyclic homology which is more accessible to techniques of homotopy theory.

The vertical maps are cyclotomic traces defined by Bökstedt-Hsiang-Madsen (1993), who first defined $TC$.

**Some Calculations**.

Simplest case: $X = *$, so $M = D^n$.

Waldhausen (1978): $A(*) \to K(\mathbb{Z})$ is a rational equivalence, hence also $\mathrm{Wh}(*) \to K(\mathbb{Z})$. Thus from known calculations in algebraic K-theory we have

$$\pi_i \mathcal{C}(D^n) \otimes \mathbb{Q} = \pi_{i+2} \mathrm{Wh}(*) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

Analogous to $\mathrm{Diff}(S^n) \simeq O(n+1) \times \mathrm{Diff}(D^n \operatorname{rel} \partial)$ one has

$$\mathrm{Diff}(D^n) \simeq O(n) \times C(D^{n-1})$$

Corollary: There are infinitely many distinct smooth fiber bundles $D^n \to E \to S^{4k}$ that are not unit disk bundles of vector bundles, when $n >> k \geq 1$. These are all topological products $S^{4k} \times D^n$ since $C_{TOP}(D^n) \simeq *$ by the Alexander trick.

From the fibration

$$\mathrm{Diff}(D^{n+1} \operatorname{rel} \partial) \to C(D^n) \to \mathrm{Diff}(D^n \operatorname{rel} \partial)$$

we conclude that either $\pi_{4k-1}\mathrm{Diff}(D^n \operatorname{rel} \partial) \otimes \mathbb{Q} \neq 0$ or $\pi_{4k-1}\mathrm{Diff}(D^{n+1} \operatorname{rel} \partial) \otimes \mathbb{Q} \neq 0$ when $n >> k$. Which one? Depends just on the parity of $n$, by:

Farrell-Hsiang (1978): In the stable range

$$\pi_i \mathrm{Diff}(D^n \operatorname{rel} \partial) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } i \equiv 3 \bmod 4 \text{ and } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Rognes (2002): Modulo odd torsion:

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_i \mathrm{Wh}(*)$ | 0 | 0 | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z} \times \mathbb{Z}_2$ | $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ |

| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|
| $\mathbb{Z}_2$ | $\mathbb{Z}_4$ | $\mathbb{Z}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_8 \times \mathbb{Z}_2$ | $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_{32} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ |

First 3-torsion is $\mathbb{Z}_3$ in $\pi_{11}\mathrm{Wh}(*)$, first 5-torsion is $\mathbb{Z}_5$ in $\pi_{18}\mathrm{Wh}(*)$.

Next step: Apply this to compute $\pi_i \mathrm{Diff}(D^n \operatorname{rel} \partial)$ for small $i << n$.

Other manifolds $M$ have been studied too, e.g., spherical (Hsiang-Jahren), Euclidean (Farrell-Hsiang), hyperbolic (Farrell-Jones) ............................

## III. Stabilization via Connected Sum.

Narrower goal: Compute $H_*(\mathrm{BDiff}(M))$. This gives characteristic classes for smooth bundles with fiber $M$.

**<u>Madsen-Weiss Theorem</u>**: Let $S_g$ be the closed orientable surface of genus $g$. Then $H_i(\mathrm{BDiff}(S_g)) \cong H_i(\Omega_0^\infty AG_{\infty,2}^+)$ for $g >> i$ (roughly $g > 3i/2$) where:

- $AG_{n,2}$ = 'affine Grassmannian' of oriented affine 2-planes in $\mathbb{R}^n$.
- $AG_{n,2}^+$ = one-point compactification of $AG_{n,2}$. (Point at $\infty$ is the empty plane.)
- $\Omega^\infty AG_{\infty,2}^+ = \cup_n \Omega^n AG_{n,2}^+$ via the natural inclusions $AG_{n,2}^+ \hookrightarrow \Omega AG_{n+1,2}^+$ translating a plane from $-\infty$ to $+\infty$ in the $(n+1)$st coordinate.
- $\Omega_0^\infty AG_{\infty,2}^+$ is one component of $\Omega^\infty AG_{\infty,2}^+$.

Remarks:

- $AG_{n,2}^+$ is the Thom space of a vector bundle over the usual Grassmannian $G_{n,2}$ of oriented 2-planes through the origin in $\mathbb{R}^n$, namely the orthogonal complement of the canonical bundle.
- Theorem usually stated in terms of mapping class groups, but the proof is via the full group $\mathrm{Diff}(S_g)$.
- Homology isomorphism but not an isomorphism on $\pi_1$. In fact the theorem can be stated as saying that the plus-construction applied to $\mathrm{BDiff}(S_\infty)$ gives $\Omega_0^\infty AG_{\infty,2}^+$.

Easy consequence (the Mumford Conjecture):

$$H_*(\mathrm{BDiff}(S_\infty); \mathbb{Q}) = \mathbb{Q}[x_2, x_4, x_6, \cdots]$$

$\mathbb{Z}_p$ coefficients much harder: Galatius 2004.

<u>Largely open problem</u>: $H_*(\mathrm{BDiff}(S_g))$ outside the stable range?

**Higher Dimensions**.

For any smooth closed (oriented) $n$-manifold there is a natural map

$$\mathrm{BDiff}(M) \to \Omega_0^\infty AG_{\infty,n}^+$$

Elements of $H^*(\Omega_0^\infty AG_{\infty,n}^+)$ pull back to characteristic classes in $H^*(\mathrm{BDiff}(M))$ that are 'universal' — independent of $M$. So one can't expect $H^*(\Omega_0^\infty AG_{\infty,n}^+)$ to give the full story on $H^*(\mathrm{BDiff}(M))$ for arbitrary $M$.

**Problem**: Find refinements of $\Omega_0^\infty AG_{\infty,n}^+$ geared toward special classes of manifolds that give analogs of the Madsen-Weiss theorem for those special classes.

Galatius, Randal-Williams (2012): Let $M_g = \#_g(S^n \times S^n)$. Then

$$H_i(\mathrm{BDiff}(M_g \,\mathrm{rel}\, D^{2n})) \cong H_i(\Omega_0^\infty \widetilde{AG}_{\infty,2n}^+) \quad \text{for } g >> i \text{ and } n > 2$$

where $\widetilde{AG}_{\infty,2n}$ denotes replacing $G_{\infty,2n}$ by its $n$-connected cover.

Again $H_*(-;\mathbb{Q})$ is easily computed to be a polynomial algebra on certain even-dimensional classes, starting in dimension $2$.

Question: Does this also work for $n = 2$? The Whitney trick works in dimension $4$ after stabilization by $\#(S^2 \times S^2)$.

**3-Manifolds**.

Two cases known:

- Let $V_g =$ standard handlebody of genus $g$. Then

$$H_i(\mathrm{BDiff}(V_g)) \cong H_i(\Omega_0^\infty S^\infty (G_{\infty,3})_+) \quad \text{for } g >> i$$

- Let $M_g = \#_g(S^1 \times S^2)$. Then

$$H_i(\mathrm{BDiff}(M_g \,\mathrm{rel}\, D^3)) \cong H_i(\Omega_0^\infty S^\infty (G_{\infty,4})_+) \quad \text{for } g >> i$$

(Homology stability proved by Chor Hang Lam, 2014.)