THE COMPLEX OF FREE FACTORS OF A FREE GROUP

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ABSTRACT. We show that the geometric realization of the partially ordered set of proper free factors in a finitely generated free group of rank n is homotopy equivalent to a wedge of spheres of dimension $n-2$.

§1. Introduction

An important tool in the study of the group $GL(n,\mathbb{Z})$ is provided by the geometric realization of the partially ordered set (poset) of proper direct summands of $\mathbb{Z}^n$. The natural inclusion $\mathbb{Z}^n \to \mathbb{Q}^n$ gives a one-to-one correspondence between proper direct summands of $\mathbb{Z}^n$ and proper subspaces of $\mathbb{Q}^n$, so that this poset is isomorphic to the spherical building X_n for $GL(n,\mathbb{Q})$. The term "spherical" comes from the Solomon-Tits theorem [8], which says that X_n has the homotopy type of a bouquet of spheres:

Solomon-Tits Theorem. *The geometric realization of the poset of proper subspaces of an n-dimensional vector space has the homotopy type of a bouquet of spheres of dimension $(n-2)$.*

The building X_n encodes the structure of parabolic subgroups of $GL(n,\mathbb{Q})$: they are the stabilizers of simplices. X_n also parametrizes the Borel-Serre boundary of the homogeneous space for $GL(n,\mathbb{R})$. The top-dimensional homology $H_{n-2}(X_n)$ is the Steinberg module I_n for $GL(n,\mathbb{Q})$, and is a dualizing module for the homology of $GL(n,\mathbb{Z})$, i.e. for all coefficient modules M there are isomorphisms

$$H^i(GL(n,\mathbb{Z});M) \to H_{d-i}(GL(n,\mathbb{Z});M\otimes I_n),$$

where $d=n(n-1)/2$ is the virtual cohomological dimension of $GL(n,\mathbb{Z})$.

If one replaces $GL(n,\mathbb{Z})$ by the group $Aut(F_n)$ of automorphisms of the free group of rank n, the natural analog FC_n of X_n is the geometric realization of the poset of proper free factors of F_n. The abelianization map $F_n \to \mathbb{Z}^n$ induces a map from FC_n to the poset of summands of $\mathbb{Z}^n$. In this paper we prove the analog of the Solomon-Tits theorem for FC_n:

Theorem 1.1. *The geometric realization of the poset of proper free factors of F_n has the homotopy type of a bouquet of spheres of dimension $n-2$.*

By analogy, we call the top homology $H_{n-2}(FC_n)$ the *Steinberg module* for $Aut(F_n)$. This leaves open some intriguing questions. It has recently been shown that $Aut(F_n)$ is a virtual duality group [1]; does the Steinberg module act as a dualizing module? There is an analog, called Autre space, of the homogeneous space for $GL(n,\mathbb{Z})$ and the Borel-Serre boundary; what is the relation between this and the "building" of free factors?

In [7], Quillen developed tools for studying the homotopy type of the geometric realization $|X|$ of a poset X. Given an order-preserving map $f\colon X\to Y$ (a "poset map"),

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there is a spectral sequence relating the homology of $|X|$, the homology of $|Y|$, and the homology of the "fibers" $|f/y|$, where

$$f/y = \{x \in X | f(x) \leq y\}$$

with the induced poset structure.

To understand FC_n then, one might try to apply Quillen's theory using the poset map $FC_n \to X_n$. However, it seems to be difficult to understand the fibers of this map. Instead, we proceed by modelling the poset of free factors topologically, as the poset B_n of simplices of a certain subcomplex of the "sphere complex" $S(M)$ studied in [2]. There is a natural poset map from B_n to FC_n; we compute the homotopy type of B_n and of the Quillen fibers of the poset map, and apply Quillen's spectral sequence to obtain the result.

§2. Sphere systems

Let M be the compact 3-manifold obtained by taking a connected sum of n copies of $S^1 \times S^2$ and removing a small open ball. A *sphere system* in M is a non-empty finite set of disjointly embedded 2-spheres in the interior of M, no two of which are isotopic, and none of which bounds a ball or is isotopic to the boundary sphere of M. The complex $S(M)$ of sphere systems in M is defined to be the simplicial complex whose k-simplices are isotopy class of sphere systems with $k+1$ spheres.

Fix a basepoint p on ∂M. The fundamental group $\pi_1(M, p)$ is isomorphic to F_n. Any automorphism of F_n can be realized by a homeomorphism of M fixing ∂M. A theorem of Laudenbach [4] implies that such a homeomorphism inducing the identity on $\pi_1(M, p)$ acts trivially on isotopy classes of sphere systems, so that in fact $Aut(F_n)$ acts on $S(M)$.

For H a subset of $\pi_1(M, p)$, define S_H to be the subcomplex of $S(M)$ consisting of isotopy classes of sphere systems S such that $\pi_1(M - S, p) \supseteq H$. Define Y_H to be the subcomplex of S_H consisting of isotopy classes of sphere systems S such that $M - S$ is connected. For H trivial, $S_H = S(M)$, and Y_H is the complex Y of [2].

The following theorems determine the homotopy types of S_H and Y_H. The proofs rely upon the proofs for $S(M)$ and Y given in [2].

Theorem 2.1. *S_H is contractible.*

Proof. We claim that the contraction of $S(M)$ in [2] restricts to a contraction of the subcomplex S_H. To show this, the key point is the following:

Lemma 2.2. *Any two simplices in S_H can be represented by sphere systems Σ and S such that every element of H is representable by a loop disjoint from both Σ and S.*

Proof. Enlarge Σ to a maximal sphere system Σ', so the components of $M - \Sigma'$ are three-punctured spheres. By Proposition 1.1 of [2] we may isotope S to be in normal form with respect to Σ'; this means that S intersects each component of $M - \Sigma'$ in a collection of surfaces, each having at most one boundary circle on each of the three punctures.

We can represent a given element of H by a loop γ_0 based at p, such that γ_0 is disjoint from S and transverse to Σ'. The points of intersection of γ_0 with Σ' divide γ_0 into a finite set of arcs, each entirely contained in one component of $M - \Sigma'$. Suppose one of these arcs α, in a component P of $M - \Sigma'$, has both endpoints on the same boundary sphere σ of P. Since the map $\pi_0(\sigma - (S \cap \sigma)) \to \pi_0(P - (S \cap P))$ is injective (an easy consequence of normal form), there is an arc α' in $\sigma - (S \cap \sigma)$ with $\partial\alpha' = \partial\alpha$. Since P is simply-connected, α is homotopic to α' fixing endpoints. This homotopy gives a homotopy of γ_0 eliminating the two points of $\partial\alpha$ from $\gamma_0 \cap \Sigma'$, without introducing any intersection points with S. After repeating this operation a finite number of times, we may assume there are no remaining arcs of $\gamma_0 - (\gamma_0 \cap \Sigma')$ of the specified sort.

Now consider a homotopy $F\colon I \times I \to M$ of γ_0 to a loop γ_1 disjoint from Σ. Make F transverse to Σ' and look at $F^{-1}(\Sigma')$. This consists of a collection of disjoint arcs and circles. These do not meet the left and right edges of $I \times I$ since these edges map to the basepoint p.

We claim that every arc component of $F^{-1}(\Sigma')$ with one endpoint on $I \times \{0\}$ must have its other endpoint on $I \times \{1\}$. If not, choose an "edgemost" arc with both endpoints on $I \times \{0\}$, i.e. an arc such that the interval of $I \times \{0\}$ bounded by the endpoints contains no other point of $F^{-1}(\Sigma')$. Then γ_0 maps this interval to an arc α in $M - S$ which is entirely contained in one component P of $M - \Sigma'$ and has both endpoints on the same boundary sphere of P, contradicting our assumption that all such arcs have been eliminated.

Since the loop γ_1 is disjoint from Σ, it follows that γ_0 must be disjoint from Σ, and by construction γ_0 was disjoint from S. $\square$

Since elements of H are representable by loops disjoint from Σ and S, these loops remain disjoint from sphere systems obtained by surgering S along Σ, because such surgery produces spheres lying in a neighborhood of $S \cup \Sigma$. This means that the contraction of S(M) constructed in [2] restricts to a contraction of the subcomplex S_H. (Alternatively, we could use the simpler contraction technique of [3], which reverses the roles of S and Σ.) $\square$

Definition. A simplicial complex K is *k-spherical* if it is k-dimensional and $(k-1)$-connected. By convention, a (-1)-*spherical* complex is a single point. A complex is *spherical* if it is k-spherical for some k.

Now that we have S_H contractible, we use it to show that Y_H is spherical. In the course of the proof, we will need to consider complexes analogous to Y_H for manifolds with more than one boundary sphere. The next lemma shows that these have the same homotopy type as Y_H.

Let M_k be the manifold obtained from the connected sum of n copies of $S^1 \times S^2$ by deleting k disjoint open balls rather than just a single ball. Choose the basepoint p on one of the spheres in ∂M. For $H \subseteq \pi_1(M_k, p)$, define $Y_H(M_k)$ to be the complex of isotopy classes of sphere systems S in M_k with connected complement such that $\pi_1(M - S, p) \supseteq H$.

Lemma 2.3. *For $k \geq 1$, $Y_H(M_{k+1})$ deformation retracts onto a subcomplex isomorphic to $Y_H(M_k)$.*

Proof. Let ∂_0 and ∂_1 be two components of ∂M, with $p \in \partial_0$. Fix a sphere σ in the interior of M_{k+1} which separates M_{k+1} into two components, one of which is a three-punctured sphere P containing ∂_0 and ∂_1, and the other of which is homeomorphic to M_k.

Let S be a sphere system representing a simplex of $Y_H(M_{k+1})$. We may assume that S is in normal form with respect to σ. This means that S intersects P in a collection of disjoint disks, each of which has boundary on σ and separates ∂_0 from ∂_1. Perform a sequence of modifications of S by transferring, one at a time, each disk of $S \cap P$ to the other side of ∂_1. This has no effect on $\pi_1(M - S, p)$, so the sphere systems which arise during this modification are still in $Y_H(M_{k+1})$. The final system resulting from transferring all the disks of $S \cap P$ to the other side of ∂_1 can be isotoped to be disjoint from σ.

This sequence of modifications can also be described as a sequence of surgeries on S using the disks in σ on the side away from ∂_0. As explained on pp. 48-49 of [2], such a surgery process defines a piecewise linear flow on the sphere complex. In the present case this flow gives a deformation retraction of $Y_H(M_{k+1})$ onto the subcomplex of sphere systems in $M - \sigma$. This subcomplex can be identified with $Y_H(M_k)$ by choosing the basepoint for M_k to be in σ. □

Theorem 2.4. *Let H be a free factor of $F_n = \pi_1(M, p)$. Then Y_H is $(n - rk(H) - 1)$-spherical.*

Proof. Let $i \leq n - rk(H) - 2$. Any map $g\colon S^i \to Y_H$ can be extended to a map $\hat{g}\colon D^{i+1} \to S_H$ since S_H is contractible. We can assume $\hat{g}$ is a simplicial map with respect to some triangulation of D^{i+1} compatible with its standard piecewise linear structure. We will redefine $\hat{g}$ on the stars of certain simplices in the interior of D^{i+1} to make the image of $\hat{g}$ lie in Y_H.

To each sphere system S we associate a dual graph $\Gamma(S)$, with one vertex for each component of $M - S$ and one edge for each sphere in S. The endpoints of the edge corresponding to $s \in S$ are the vertices corresponding to the component or components adjacent to s. We say a sphere system S is *purely separating* if $\Gamma(S)$ has no edges which begin and end at the same vertex. Each sphere system S has a *purely separating core*, consisting of those spheres in S which correspond to the core of $\Gamma(S)$, i.e. the subgraph spanned by edges with distinct vertices. The purely separating core of $S \in S_H$ is empty if and only if S is in Y_H.

Let σ be a simplex of D^{i+1} of maximal dimension among the simplices τ with $\hat{g}(\tau)$ purely separating. Note that all such simplices τ lie in the interior of D^{i+1}, since the boundary of D^{i+1} maps to Y_H. Let $S = \hat{g}(\sigma)$, and let $N_0, \ldots, N_r$ $(r \geq 1)$ be the connected components of $M - S$, with $p \in N_0$. A simplex τ in the link $lk(\sigma)$ maps to a system T in the link of S, so that each $T_j = T \cap N_j$ is a sphere system in N_j and $H \leq \pi_1(N_0 - T_0, p)$. Furthermore $N_j - T_j$ must be connected for all j since otherwise the core of $\Gamma(S \cup T)$ would have more edges than $\Gamma(S)$, contradicting the maximality of σ. Thus $\hat{g}$ maps $lk(\sigma)$ into a subcomplex of S_H which can be identified with $Y_H(N_0) * Y(N_1) * \cdots * Y(N_r)$.

Since σ is a simplex in the interior of D^{i+1}, $lk(\sigma)$ is a sphere of dimension $i - dim(\sigma)$. Each N_j has fundamental group of rank $n_j < n$, so by Lemma 2.3 and induction, $Y_H(N_0)$ is $(n_0 - rk(H) - 1)$-spherical and, for $j \geq 1$, $Y(N_j)$ is $(n_j - 1)$-spherical. Hence $Y_H(N_0) * Y(N_1) * \cdots * Y(N_r)$ is spherical of dimension $(\sum_{j=0}^{r} n_j) - rk(H) - 1$.

Now $n = (\sum_j n_j) + rk(\pi_1(\Gamma(S))) = (\sum_j n_j) + m - r$ where m is the number of spheres in S, i.e., edges in $\Gamma(S)$. Since a simplicial map cannot increase dimension, we have $\dim(\sigma) \geq m - 1$. Since $i \leq n - rk(H) - 2$, we have

$$\begin{aligned} i - \dim(\sigma) &\leq n - rk(H) - 2 - \dim(\sigma) \\ &\leq n - rk(H) - m - 1 \\ &= \Big(\sum_j n_j\Big) - rk(H) - 1 - r \\ &< \Big(\sum_j n_j\Big) - rk(H) - 1. \end{aligned}$$

Hence the map $\hat{g}\colon lk(\sigma) \to Y_H(N_0) * Y(N_1) * \cdots * Y(N_r)$ can be extended to a map of a disk D^k into $Y_H(N_0) * Y(N_1) * \cdots * Y(N_r)$, where $k = i + 1 - dim(\sigma)$. The system S is compatible with every system in the image of D^k, so this map can be extended to a map $\sigma * D^k \to S_H$. We replace the star of σ in D^{i+1} by the disk $\partial(\sigma) * D^k$, and define $\hat{g}$ on $\partial(\sigma) * D^k$ using this map.

What have we improved? The new simplices in the disk $\partial(\sigma) * D^k$ are of the form $\sigma' * \tau$, where σ' is a face of σ and $\hat{g}(\tau) \subset Y_H(N_0) * Y(N_1) * \cdots * Y(N_r)$. The image of such a simplex $\sigma' * \tau$ is a system $S' \cup T$ such that in $\Gamma(S' \cup T)$ the edges corresponding to T are all loops. Therefore any simplex in the disk $\partial(\sigma) * D^k$ with purely separating image must lie in the boundary of this disk, where we have not modified $\hat{g}$.

We continue this process, eliminating purely separating simplices until there are none in the image of $\hat{g}$. Since every system in $S_H - Y_H$ has a non-trivial purely separating core, in fact the whole disk maps into Y_H, and we are done. □

§3. Free factors

We now turn to the poset FC_n of proper free factors of the free group F_n, partially ordered by inclusion. A k-simplex in the geometric realization $|FC_n|$ is a flag $H_0 < H_1 < \ldots < H_k$ of proper free factors of F_n, each properly included in the next. Each H_i is also a free factor of H_{i+1} (see [6, p. 117]), so that a maximal simplex of $|FC_n|$ has dimension $n-2$.

We want to model free factors of F_n by sphere systems in $Y = Y(M)$, by taking the fundamental group of the (connected) complement. A sphere system with n spheres and connected complement, corresponding to an $(n-1)$-dimensional simplex of Y, in fact has simply-connected complement. But we only want to consider proper free factors, so instead we consider the $(n-2)$-skeleton $Y^{(n-2)}$. Since Y is $(n-2)$-connected by Theorem 2.4, $Y^{(n-2)}$ is $(n-2)$-spherical.

In order to relate $Y^{(n-2)}$ to FC_n, we take the barycentric subdivision B_n of $Y^{(n-2)}$. Then B_n is the geometric realization of a poset of isotopy classes of sphere systems, partially ordered by inclusion. If $S \subseteq S'$ are sphere systems, we have $\pi_1(M - S, p) \geq \pi_1(M - S', p)$, reversing the partial ordering. Taking fundamental group of the complement thus gives a poset map $f\colon B_n \to (FC_n)^{op}$, where $(FC_n)^{op}$ denotes FC_n with the opposite partial ordering.

Proposition 3.1. *$f\colon B_n \to (FC_n)^{op}$ is surjective.*

Proof. Every simplex of FC_n is contained in a simplex of dimension $n-2$ so it suffices to show f maps onto all $(n-2)$-simplices. The group $Aut(F_n)$ acts transitively on $(n-2)$-simplices of FC_n, and all elements of $Aut(F_n)$ are realized by homeomorphisms of M, so f will be surjective if its image contains a single $(n-2)$-simplex, which it obviously does. □

Corollary 3.2. *FC_n is connected if $n \geq 3$.*

Proof. Theorem 2.4 implies that B_n is connected for $n \geq 3$. So, given any two vertices of FC_n, lift them to vertices of B_n by Proposition 3.1, connect the lifted vertices by a path, then project the path back down to FC_n. □

For any proper free factor H, let $B_{\geq H}$ denote the fiber f/H, consisting of isotopy classes of sphere systems S in B_n with $\pi_1(M-S,p) \geq H$.

Proposition 3.3. *Let H be a proper free factor of F_n. Then $B_{\geq H}$ is $(n-rk(H)-1)$-spherical. If $rk(H) = n-1$ then $B_{\geq H}$ is a single point.*

Proof. The fiber $B_{\geq H}$ is the barycentric subdivision of Y_H, so is $(n-rk(H)-1)$-spherical by Theorem 2.4.

Suppose $rk(H) = n-1$, so that $\pi_1(M,p) = H * \langle x \rangle$ for some x. An element of $B_{\geq H}$ is represented by a sphere system containing exactly one sphere s, which is non-separating with $\pi_1(M-s,p) = H$. Suppose s and s' are two such spheres. Since s and s' are both non-separating, there is a homeomorphism h of M taking s to s'. Since automorphisms of H can be realized by homeomorphisms of M fixing s, we may assume that the induced map on $\pi_1(M,p)$ is the identity on H.

Claim. The induced map $h_*\colon \pi_1(M,p) \to \pi_1(M,p)$ must send x to an element of the form $Ux^{\pm 1}V$, with $U, V \in H$.

Proof. Let $\{x_1, \ldots, x_{n-1}\}$ be a basis for H, and let W be the the reduced word representing $h_*(x)$ in the basis $\{x_1, \ldots, x_{n-1}, x\}$ for $\pi_1(M,p)$. By looking at the map induced by h on homology, we see that the exponent sum of x in W must be ± 1. Since h_* fixes H, $\{x_1, \ldots, x_{n-1}, W\}$ is a basis for $\pi_1(M,p)$. If W contained both x and x^{-1}, we could apply Nielsen automorphisms to the set $\{x_1, \ldots, x_{n-1}, W\}$ until W was of the form $x^{\pm 1}W_0x^{\pm 1}$. But $\{x_1, \ldots, x_{n-1}, x^{\pm 1}W_0x^{\pm 1}\}$ is not a basis, since it is Nielsen reduced and not of the form $\{x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}, x^{\pm 1}\}$ (see [5], Prop. 2.8)).

The automorphism fixing H and sending $x \mapsto Ux^{\pm 1}V$ can be realized by a homeomorphism h' of M which takes s to itself (see [4], Lemme 4.3.1). The composition $h'h^{-1}$ sends s' to s and induces the identity on π_1, hence acts trivially on the sphere complex. Thus s and s' are isotopic. □

Corollary 3.4. *FC_n is simply connected for $n \geq 4$.*

Proof. Let $e_0, e_1, \ldots, e_k$ be the edges of an edge-path loop in FC_n, and choose lifts $\tilde{e}_i$ of these edges to B_n. Let $e_{i-1}e_i$ or e_ke_0 be two adjacent edges of the path, meeting at the vertex H. The lifts $\tilde{e}_{i-1}$ and $\tilde{e}_i$ may not be connected, i.e. $\tilde{e}_{i-1}$ may terminate at a sphere system S' and $\tilde{e}_i$ may begin at a different sphere system S. However, both S and S' are in the fiber $B_{\geq H}$, which is connected by Proposition 3.3, so we may connect S and S' by a path in $B_{\geq H}$. Connecting the endpoints of each lifted edge in this way, we obtain a loop in B_n, which may be filled in by a disk if $n \geq 4$, by Proposition 3.3. The projection of this loop to FC_n is homotopic to the original loop, since each extra edge-path segment we added projects to a loop in the star of some vertex H, which is contractible. Therefore the projection of the disk kills our original loop in the fundamental group. □

Remark. It is possible to describe the complex Y_H purely in terms of F_n. Suppose first that H is trivial. Define a simplicial complex Z to have vertices the rank $n-1$ free factors of F_n, with a set of k such factors spanning a simplex in Z if there is an automorphism of F_n taking them to the k factors obtained by deleting the standard basis elements $x_1, \cdots, x_k$ of F_n one at a time. There is a simplicial map $f\colon Y \to Z$ sending a system of k spheres to the set of k fundamental groups of the complements of these spheres. These fundamental groups are equivalent to the standard set of k rank $n-1$ factors under an automorphism of F_n since the homeomorphism group of M acts transitively on simplices of Y of a given dimension, and the standard k factors are the fundamental groups of the complements of the spheres in a standard system in Y. The last statement of Proposition 3.3 says that f is a bijection on vertices, so f embeds Y as a subcomplex of Z. The maximal simplices in Y and Z have dimension $n-1$ and the groups $Homeo(M)$ and $Aut(F_n)$ act transitively on these simplices, so f must be surjective, hence an isomorphism. When H is nontrivial, f restricts to an isomorphism from Y_H onto the subcomplex Z_H spanned by the vertices which are free factors containing H.

We are now ready to apply Quillen's spectral sequence to compute the homology of B_n and thus prove the main theorem.

Theorem 3.5. *FC_n is $(n-2)$-spherical.*

Proof. We prove the theorem by induction on n. If $n \leq 4$, the theorem follows from Corollaries 3.2 and 3.4.

Quillen's spectral sequence [7, 7.7] applied to $f\colon B_n \to (FC_n)^{op}$ becomes

$$E^2_{p,q} = H_p(FC_n; H \mapsto H_q(B_{\geq H})) \Rightarrow H_{p+q}(B_n),$$

where the E^2-term is computed using homology with coefficients in the functor $H \mapsto H_q(B_{\geq H})$.

For $q = 0$, Corollary 3.2 gives $H_0(B_{\geq H}) = \mathbb{Z}$ for all H, so $E^2_{p,0} = H_p(FC_n, \mathbb{Z})$.

For $q > 0$, we have $E^2_{p,q} = H_p(FC_n; H \mapsto \widetilde{H}_q(B_{\geq H}))$, and we follow Quillen ([7], proof of Theorem 9.1) to compute this.

For a subposet A of FC_n, let L_A denote the functor sending H to a fixed abelian group L if $H \in A$ and to 0 otherwise. Set $U = FC_{\leq H} = \{H' \in FC_n | H' \leq H\}$ and $V = FC_{<H} = \{H' \in FC_n | H' < H\}$. Then

$$H_i(FC_n; L_V) = H_i(V; L), \text{ and}$$
$$H_i(FC_n; L_U) = H_i(U; L) = \begin{cases} L & \text{if } i = 0 \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

since $|U|$ is contractible. The short exact sequence of functors

$$1 \to L_V \to L_U \to L_{\{H\}} \to 1$$

gives a long exact homology sequence, from which we compute

$$H_i(FC_n; L_{\{H\}}) = \widetilde{H}_{i-1}(FC_{<H}; L).$$

Now, $H \mapsto \widetilde{H}_q(B_{\geq H})$ is equal to the functor

$$\bigoplus_{rk(H)=n-q-1} \widetilde{H}_q(B_{\geq H})_{\{H\}}$$

since $B_{\geq H}$ is $(n - rk(H) - 1)$-spherical by Proposition 3.3. Thus

$$\begin{aligned} E^2_{p,q} &= H_p(FC_n; H \mapsto \widetilde{H}_q(B_{\geq H})) \\ &= \bigoplus_{rk(H)=n-q-1} H_p(FC_n; \widetilde{H}_q(B_{\geq H})_{\{H\}}) \\ &= \bigoplus_{rk(H)=n-q-1} \widetilde{H}_{p-1}(FC_{<H}; \widetilde{H}_q(B_{\geq H})) \end{aligned}$$

Free factors of F_n contained in H are also free factors of H. Since H has rank $< n$. $FC_{<H}$ is $(rk(H)-2)$-spherical by induction. Therefore $E^2_{p,q} = 0$ unless $p-1 = (n-q-1)-2$, i.e. $p+q = n-2$. Since all terms in the E^2-term of the spectral sequence are zero except the bottom row for $p \leq n-2$ and the diagonal $p+q = n-2$, all differentials are zero and we have $E^2 = E^\infty$ as in the following diagram:

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & & & \\
E^2_{0,n-2} & 0 & 0 & 0 & & & \\
0 & E^2_{1,n-3} & 0 & 0 & & & \\
0 & 0 & E^2_{2,n-4} & 0 & & & \\
0 & 0 & 0 & E^2_{3,n-5} & & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
H_0(FC_n) & H_1(FC_n) & H_2(FC_n) & H_3(FC_n) & \cdots & H_{n-2}(FC_n) & 0
\end{array}$$

Since FC_n is connected and the spectral sequence converges to $H_*(B_n)$, which is $(n-3)$-connected, we must have $\tilde{H}_i(FC_n) = 0$ for $i \neq n-2$. Since FC_n is simply-connected by Corollary 3.4, this implies that FC_n is $(n-3)$-connected by the Hurewicz theorem. The theorem follows since FC_n is $(n-2)$-dimensional. $\square$

§4. The Cohen-Macaulay Property

In a PL triangulation of an n-dimensional sphere, the link of every k-simplex is an $(n-k-1)$-sphere. A poset is said to be *Cohen-Macaulay* of dimension n if its geometric realization is n-spherical and the link of every k-simplex is $(n-k-1)$-spherical (see [7]). Spherical buildings are Cohen-Macaulay, and we remark that FC_n also has this nice local property.

To see this, let $\sigma = H_0 < H_1 < \ldots < H_k$ be a k-simplex of FC_n. The link of σ is the join of subcomplexes $FC_{H_i,H_{i+1}}$ of FC_n spanned by free factors H with $H_i < H < H_{i+1}$ ($-1 \leq i \leq k$, with the conventions $H_{-1} = 1$ and $H_{k+1} = F_n$). Counting dimensions, we see that it suffices to show that for each r and s with $0 \leq s < r$, the poset $FC_{r,s}$ of proper free factors H of F_r which properly contain F_s is $(r-s-2)$-spherical. The proof of this is identical to the proof that FC_n is $(n-2)$-spherical, after setting $n = r$ and replacing the complex Y by Y_{F_s}.

§5. The map to the building.

As mentioned in the introduction, the abelianization map $F_n \to \mathbb{Z}^n$ induces a map from the free factor complex FC_n to the building X_n, since summands of $\mathbb{Z}^n$ correspond to subspaces of $\mathbb{Q}^n$. Since the map $Aut(F_n) \to GL(n, \mathbb{Z})$ is surjective, every basis for $\mathbb{Z}^n$ lifts to a basis for F_n, and hence every flag of summands of $\mathbb{Z}^n$ lifts to a flag of free factors of F_n, i.e. the map $FC_n \to X_n$ is surjective.

Given a basis $\{v_1, \ldots, v_n\}$ for $\mathbb{Q}^n$, consider the subcomplex of X_n consisting of all flags of subspaces of the form $\langle v_{i_1} \rangle \subset \langle v_{i_1}, v_{i_2} \rangle \subset \ldots \subset \langle v_{i_1}, \ldots, v_{i_{n-1}} \rangle$. This subcomplex can be identified with the barycentric subdivision of the boundary of an $(n-1)$-dimensional simplex, so forms an $(n-2)$-dimensional sphere in X_n, called an *apartment*.

The construction above applied to a basis of F_n instead of $\mathbb{Q}^n$ yields an $(n-2)$-dimensional sphere in FC_n. In particular, $H_{n-2}(FC_n)$ is non-trivial. This sphere maps to an apartment in X_n showing that the induced map $H_{n-2}(FC_n) \to H_{n-2}(X_n)$ is also non-trivial.

The property of buildings which is missing in FC_n is that given any two maximal simplices there is an apartment which contains both of them. For example, for $n = 3$ and F_3 free on $\{x, y, z\}$ there is no "apartment" which contains both the one-cells corresponding to $\langle x \rangle \subset \langle x, y \rangle$ and $\langle yxy^{-1} \rangle \subset \langle x, y \rangle$, since x and yxy^{-1} do not form part of a basis of F_3.

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