Boundary Curves of Incompressible Surfaces

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This is a Tex version, made in 2004, of a paper that appeared in Pac. J. Math. 99 (1982), 373-377, with some revisions in the exposition.

Let M be a compact orientable 3-manifold whose boundary ∂M consists of a single torus. If a meridian and longitude in this torus are chosen, then isotopy classes of smoothly embedded circles in ∂M that do not bound disks in ∂M correspond bijectively with elements of $\mathbb{Q}P^1 = \mathbb{Q} \cup \{1/0\}$, regarded as slopes of these curves. We show in this paper that the set of slopes coming from boundary curves of incompressible, ∂-incompressible surfaces in M is finite.

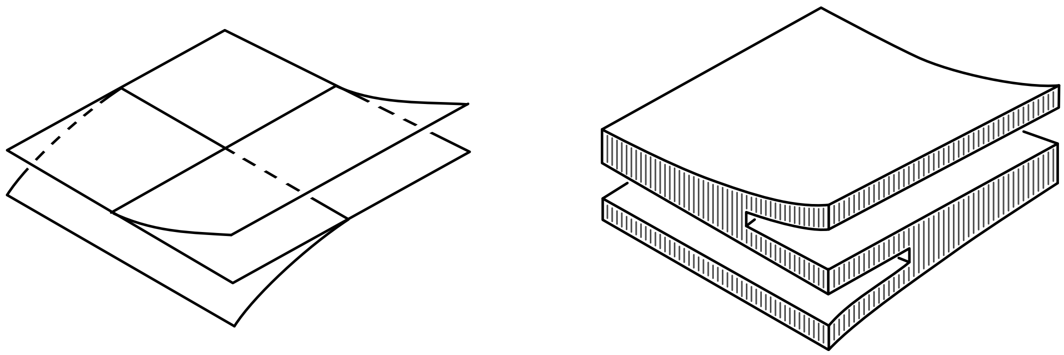
There is also a generalization to the case that ∂M consists of n tori $T_1, \cdots, T_n$. Given a curve system in ∂M consisting of finitely many disjoint smoothly embedded circles that do not bound disks in ∂M, then by choosing parallel orientations for the circles in each component of ∂M, we get an element of $H_1(\partial M)$. Ignoring orientations amounts to factoring out multiplication by ± 1 in each component, yielding a quotient of $H_1(\partial M)$ which can be identified with the set $CS(\partial M)$ of isotopy classes of curve systems in ∂M. Each factor $H_1(T_i)$ of $H_1(\partial M)$ is the integer lattice in $H_1(T_i;\mathbb{R}) \approx \mathbb{R}^2$, and $H_1(T_i;\mathbb{R})/\pm 1$ is a cone, so $CS(\partial M)$ can be viewed as the integer lattice in a product of cones, the space $H_1(\partial M;\mathbb{R})/\mathbb{Z}_2^n$ where $\mathbb{Z}_2^n$ acts by inversions in the factors $H_1(T_i;\mathbb{R})$.

Theorem 1. *The subset of $CS(\partial M)$ consisting of curve systems that bound incompressible, ∂-incompressible surfaces in M is contained in the image of the union of a finite number of rank n subgroups of $H_1(\partial M) \approx \mathbb{Z}^{2n}$.*

Corollary. *In the case that ∂M is a single torus, there are just finitely many slopes realized by boundary curves of incompressible, ∂-incompressible surfaces in M.*

One may compare the assertion of the theorem with the fact (a consequence of duality) that the image of the boundary map $H_2(M,\partial M) \to H_1(\partial M)$ has rank equal to one-half the rank of $H_1(\partial M)$. This image will be one of the rank n subgroups of $H_1(\partial M)$ referred to in the theorem since all elements of $H_2(M, \partial M)$ are represented by incompressible, ∂-incompressible surfaces.

The proof of the theorem will follow fairly easily from a fundamental result of [FO] about branched surfaces in 3-manifolds, which are closed subsets locally diffeomorphic to the model in the first figure below.



A branched surface B is said to *carry* a surface S if S lies in a fibered regular neighborhood $N(B)$ of B, indicated in the second figure, and is transverse to all the fibers of $N(B)$. If S meets all the fibers of $N(B)$ it is said to have *positive weights*. The result of [FO] is that in a compact irreducible 3-manifold M with incompressible boundary there exist finitely many branched surfaces $(B_i, \partial B_i) \subset (M, \partial M)$ such that the surfaces carried with positive weights by these B_i's are exactly all the incompressible, ∂-incompressible surfaces in M, up to isotopy. A refinement in [O] is that the B_i's can be chosen so that all the surfaces they carry, whether of positive weights or not, are incompressible and ∂-incompressible.

Let B be one of these branched surfaces B_i. Then $\partial B = B \cap \partial M$ is a train track, or branched 1-manifold, in ∂M with two key properties:

(1) There is no smooth disk $D \subset \partial M$ with $D \cap \partial B = \partial D$.

(2) There is no disk $D \subset \partial M$, smooth except for one outward cusp point in ∂D, such that $D \cap \partial B = \partial D$.

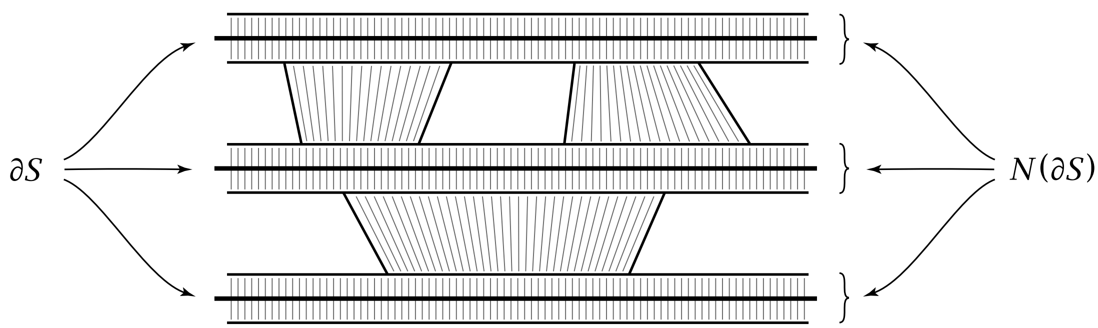
The latter condition is explicitly given in [FO]. If condition (1) failed, then any surface carried by B with positive weights would have a boundary circle which was contractible in ∂M. By incompressibility, this circle would bound a disk component of the surface, contrary to the construction of B in [FO]. Condition (2) can be phrased as saying that the train track ∂B has no monogons. Sometimes train tracks are required to have no digons as well, but we have to allow these here.

Let S be a surface carried by B with positive weights. No component of ∂S can be contractible in ∂M, since otherwise there would be a smooth disk $D \subset \partial M$ with $\partial D \subset \partial B$, and somewhere inside this disk condition (1) or (2) would be violated. Thus in each component torus T_i of ∂M which B meets, ∂S consists of a number of parallel circles.

To simplify notation in what follows, if T_i is one of the tori of ∂M we let $\partial_i B = \partial B \cap T_i$, and similarly $\partial_i S = \partial S \cap T_i$ for any surface S carried by B.

Lemma 1. *There is an orientation ω of ∂B such that, for each surface S carried by B and each torus T_i of ∂M, all the circles of $\partial_i S$, with the orientations induced from ω, are homologous in T_i.*

Proof: Choose a surface S carried by B with positive weights. We can construct a fibered regular neighborhood $N(\partial S)$ of ∂S in ∂M from a fibered regular neighborhood $N(\partial B)$ of ∂B in ∂M by removing certain fibered rectangles and annuli. Inverting this process, we can build $N(\partial B)$ from $N(\partial S)$ by adding fibered rectangles and annuli.



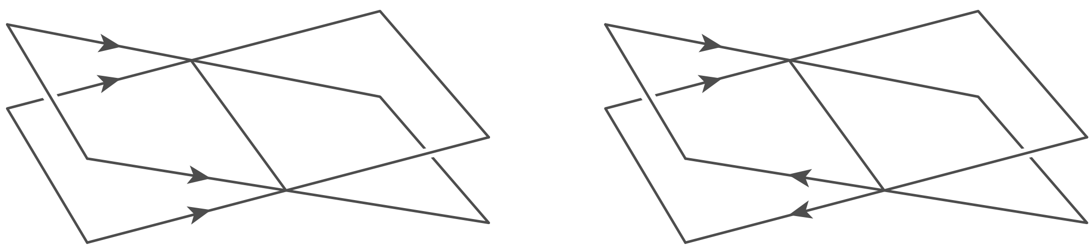
No fiber of an added rectangle can join a component of $N(\partial S)$ to itself, otherwise condition (2) would be violated. This implies in particular that all the complementary regions of $N(\partial B)$ in each T_i are rectangles or annuli. If we choose parallel orientations for all the circles of $\partial_i S$, this determines an orientation ω_i for $\partial_i B$ which induces the chosen orientation of $\partial_i S$. An orientation for T_i then gives an orientation for all the fibers of $N(\partial_i B)$.

We may choose an oriented simple closed curve γ_i in T_i meeting $N(\partial_i B)$ in a union of fibers, such that the orientation of γ_i agrees with the orientation of the fibers. To do this, start with any fiber of $N(\partial_i B)$, continue across a complementary rectangle or annulus to another fiber of $N(\partial_i B)$ on the opposite side of this rectangle or annulus, and so on. Eventually the curve so constructed must either close up or come arbitrarily close to closing up, in which case by rechoosing a part of the curve in one of the complementary rectangles or annuli we can make it close up.

The statement of the lemma now follows from the existence of γ_i, since for an arbitrary surface S carried by B, if we orient $\partial_i S$ via ω_i, then all the points of $\gamma_i \cap \partial_i S$ have intersection numbers of the same sign. □

Lemma 2. *Let S_1 and S_2 be surfaces carried by B, with boundaries oriented as in Lemma 1. Then the algebraic inersection number $\partial S_1 \cdot \partial S_2$, computed using an orientation of ∂M as the boundary of M, is zero.*

Proof: Perturb S_1 and S_2 slightly to be transverse and still transverse to fibers of $N(B)$. There are two possible configurations for the orientations of ∂S_1 and ∂S_2 at the ends of an arc α of $S_1 \cap S_2$, shown in the following figure, where the fibers of $N(B)$ are vertical:



In both cases the intersection numbers ± 1 at the two ends of α have opposite sign. Thus all points of $\partial S_1 \cap \partial S_2$ cancel algebraically in pairs. □

Proof of the Theorem: Let us first do the simpler case that ∂M consists of a single torus. In this case Lemma 2 implies that for any two surfaces carried by B, the boundary circles of one surface have the same slope as the boundary circles of the other surface. Thus each of the finitely choices for B has a unique boundary slope for the surfaces it carries, so there are only finitely many boundary slopes in all.

Now consider the general case that ∂M has n boundary tori. Orienting the boundary curves of surfaces carried by B as in Lemma 1, these curves generate a subgroup of $H_1(\partial M)$ on which the intersection form is identically zero by Lemma 2. Passing to real coefficients, the subspace of $H_1(\partial M; \mathbb{R})$ spanned by these boundary surfaces then has dimension at most n by a standard elementary linear algebra argument. So the subgroup of $H_1(\partial M)$ generated by the boundary curves for B has rank at most n. As there are only finitely many choices for B, the result follows. □

One can be more precise about the global structure of the subset $BCS(M)$ of $CS(\partial M)$ consisting of curve systems that bound incompressible surfaces. For B one of the branched surfaces B_i considered above, the surfaces carried by B are determined by assigning nonnegative integer weights a_i to the components of $B - B'$, where B'

is the branching locus of B. These weights must satisfy certain equations of the form $a_i + a_j = a_k$ coming from the branching of B, and every set of weights satisfying these equations gives rise to a surface carried by B. Thus if $B - B'$ has N components, the surfaces carried by B correspond to the integer points of a convex polyhedral cone c_B in $\mathbb{R}^N$ which is the intersection of the orthant $[0,\infty)^N$ with the linear subspace of $\mathbb{R}^N$ defined by the branch equations.

After choosing orientations for the train tracks $\partial_i B$ as in Lemma 1, then by taking boundary curves we obtain a linear map $c_B \to H_1(\partial M;\mathbb{R}) = \mathbb{R}^{2n}$. The image of this map is another polyhedral cone, and the union of these images, for varying orientations on ∂B and different choices of B, is a conical polyhedral complex. This projects onto another conical polyhedral complex in $H_1(\partial M;\mathbb{R})/\mathbb{Z}_2^n$ whose integer points are $BCS(M)$. The theorem implies that this complex $X(M)$ has dimension at most n, one-half the dimension of the ambient space $H_1(\partial M;\mathbb{R})/\mathbb{Z}_2^n$. Since the branching equations have integer coefficients, the rays from the origin through integer points of $X(M)$ are dense in $X(M)$. We can then projectivize by passing to the space of these rays, and the projective classes of bounding curve systems will be dense in this projectivization $PX(M)$ of $X(M)$. The corresponding projectivization of $H_1(\partial M;\mathbb{R})/\mathbb{Z}_2^n$ is the projective lamination space $PL(\partial M)$, a sphere S^{2n-1} containing the polyhedral complex $PX(M)$.

The original version of this paper concluded with three questions which have subsequently been answered:

(1) Is there a generalization of the theorem to 3-manifolds having boundary components of higher genus? This was done in [F].

(2) For knot exteriors in S^3, must the boundary slopes always be integers? This was asked because at the time the only examples that had been computed had this property, but in [HO] it was shown that every rational number occurs as a boundary slope for some Montesinos knot.

(3) Are there nontrivial knots having only one boundary slope? It was shown in [MS] that the answer is no. But there are knots with only two boundary slopes, namely torus knots.

References

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