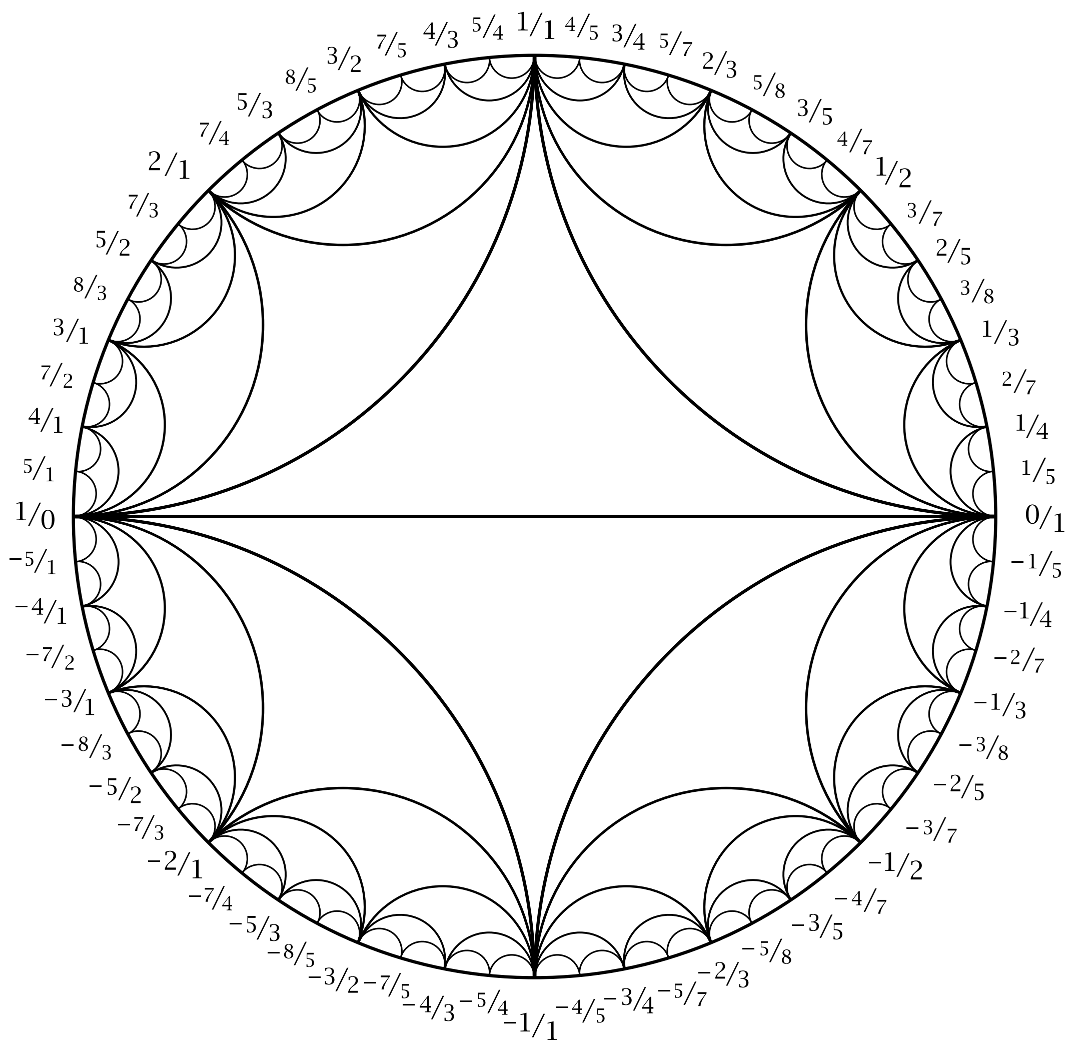
1 The Farey Diagram

Our goal is to use geometry to study numbers. Of the various kinds of numbers, the simplest are integers, along with their ratios, the rational numbers. Usually one thinks of rational numbers geometrically as points along a line, interspersed with irrational numbers as well. In this chapter we introduce a two-dimensional pictorial representation of rational numbers that displays certain interesting relations between them that we will be exploring. This diagram, along with several variants of it that will be introduced later, is known as the *Farey diagram*. The origin of the name will be explained when we get to one of these variants. Here is the diagram:

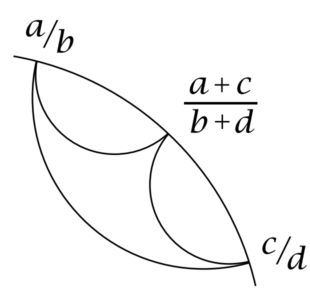


What is shown here is not the whole diagram but only a finite part of it. The actual diagram has infinitely many curvilinear triangles, getting smaller and smaller out near the boundary circle. The diagram can be constructed by first inscribing the two big triangles in the circle, then adding the four triangles that share an edge with the two big triangles, then the eight triangles sharing an edge with these four, then sixteen more triangles, and so on forever. With a little practice one can draw the diagram without lifting one's pencil from the paper: First draw the outer circle starting at the left or right side, then the diameter, then make the two large triangles, then the four next-largest triangles, and so on.

Our first task will be to explain how the vertices of all the triangles are labeled with rational numbers. Perhaps the reader can guess what the rules are before we spell them out in detail.

1.1 The Mediant Rule

The vertices of the triangles in the Farey diagram are labeled with fractions a/b, including the fraction $1/0$ for ∞, according to the following scheme. In the upper half of the diagram, first label the vertices of the big triangle $1/0$, $0/1$, and $1/1$. Then add labels for successively smaller triangles by the rule that, if the labels at the two ends of the long edge of a triangle are a/b and c/d, then the label on the third vertex of the triangle is $a+c/b+d$, so the numerators and denominators are added separately, contrary to the usual way of adding fractions. The fraction $a+c/b+d$ is called the *mediant* of a/b and c/d.

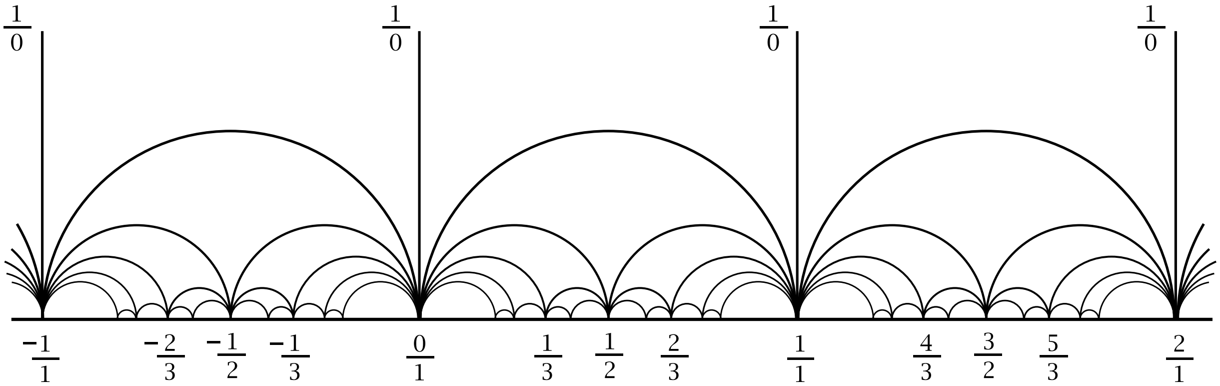


The labels in the lower half of the diagram follow the same scheme, starting with the labels $-1/0$, $0/1$, and $-1/1$ on the large triangle. Using $-1/0$ instead of $1/0$ as the label of the vertex at the far left means that we are regarding $+\infty$ and $-\infty$ as the same. The labels in the lower half of the diagram are the negatives of those in the upper half, and the labels in the left half are the reciprocals of those in the right half.

For fractions with a nonzero denominator our usual rule will be to write them with a positive denominator, so the sign of the fraction is the sign of the numerator.

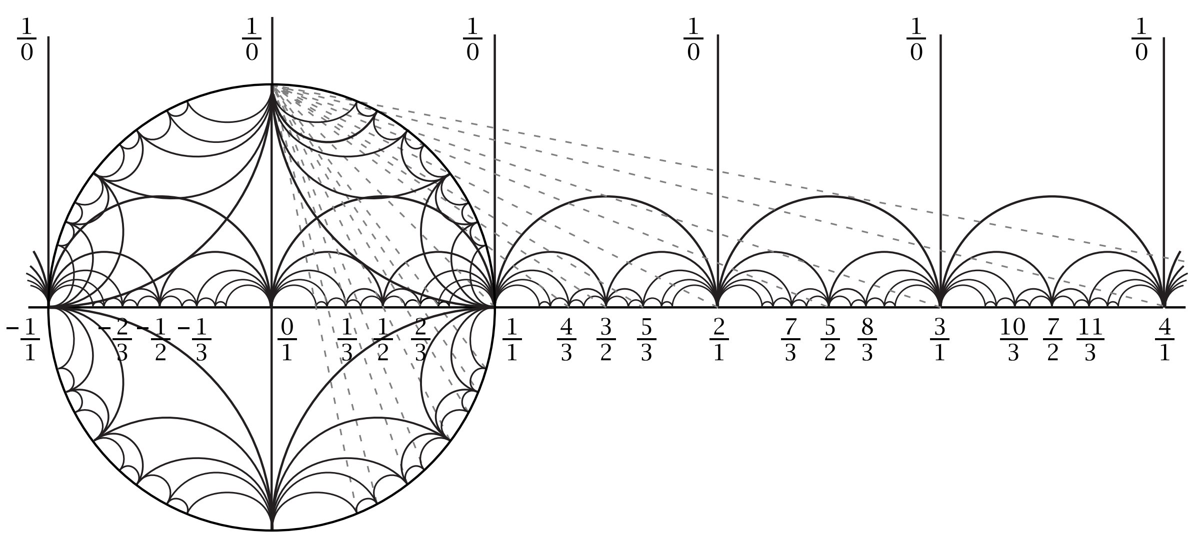
The labels generated by the mediant rule occur in their proper order around the circle, increasing from $-\infty$ to $+\infty$ as one goes around the circle in the counterclockwise direction. This is obviously true for the integer labels, and to verify it for the others it suffices to show that the mediant $a+c/b+d$ of a/b and c/d is always a number between a/b and c/d (hence the term "mediant"). Thus we want to show that if $a/b < c/d$ then $a/b < a+c/b+d < c/d$. These fractions all have positive denominators, so the inequality $a/b < c/d$ is equivalent to $ad < bc$ and $a/b < a+c/b+d$ is equivalent to $ab + ad < ab + bc$. Obviously $ad < bc$ implies $ab + ad < ab + bc$, so $a/b < c/d$ implies $a/b < a+c/b+d$. Similarly $a+c/b+d < c/d$ is equivalent to $ad + cd < bc + cd$ which also follows from $ad < bc$, so $a/b < c/d$ implies $a+c/b+d < c/d$.

There is another version of the Farey diagram with the boundary circle straightened out to a line:



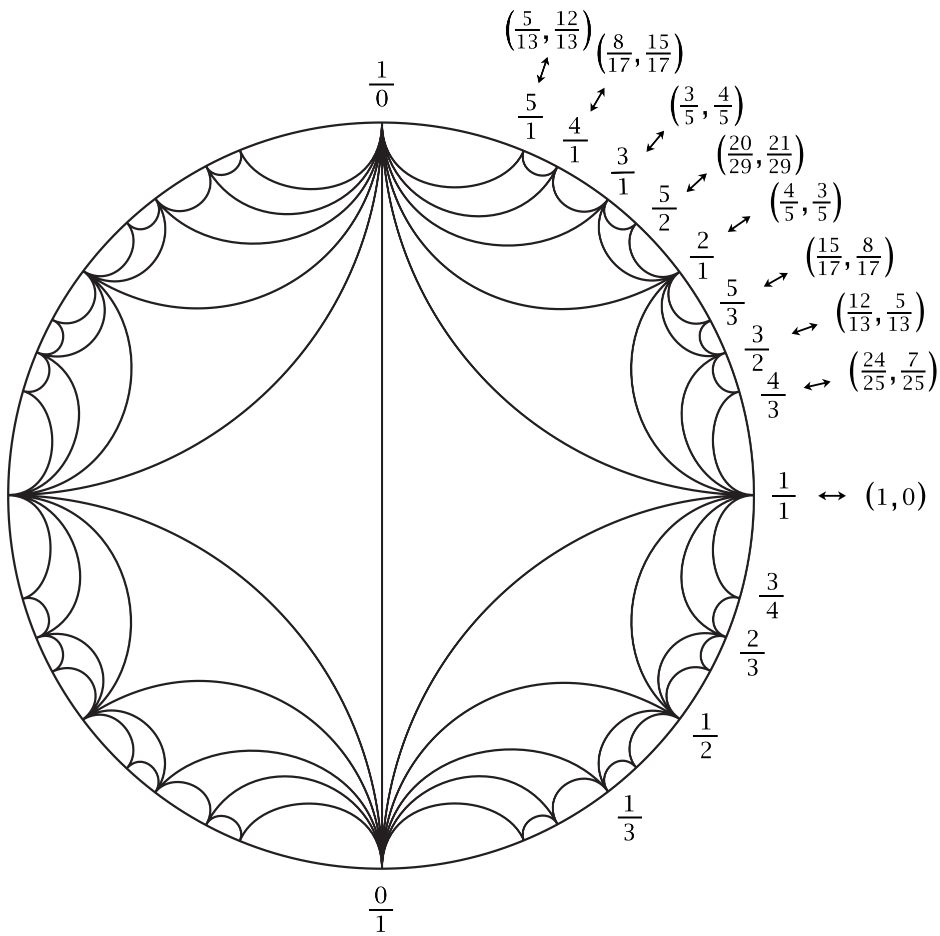
Here the diagram fills up the upper half of the xy-plane, with the vertex $^1/_0$ of the original Farey diagram positioned "at infinity" so it is not actually shown in the new version. The edges of the diagram with one endpoint at $^1/_0$ are drawn as vertical lines with lower endpoints at the integer points on the x-axis. All the other edges of the diagram are semicircles with endpoints on the x-axis, and we can position these so that the vertex labeled $^a/_b$ is actually the number $^a/_b$ on the x-axis. This is possible since when we construct the diagram by adding more and more curvilinear triangles, we can place the new vertex of each triangle at any point between its outer two vertices, so we just choose this new vertex to be at the mediant of the outer two vertices. With this rule the part of the diagram between each pair of consecutive integers n and $n+1$ looks the same since the mediant of $n + {}^a/_b$ and $n + {}^c/_d$ is $n + {}^{a+c}/_{b+d}$ as one can easily check by a simple calculation.

In the previous chapter we described how rational points (x, y) on the unit circle $x^2 + y^2 = 1$ correspond to rational points $^p/_q$ on the x-axis by means of lines through the point $(0, 1)$ on the circle. Using this correspondence, we can label the rational points on the circle by the corresponding rational points on the x-axis and then construct a new Farey diagram in the circle by filling in triangles by the mediant rule just as before.



This gives a version of the circular Farey diagram that is rotated by 90 degrees to put $1/0$ at the top of the circle, and there are also some perturbations of the positions of the other vertices and the shapes of the triangles. For our purposes these perturbations will not make much of a difference since it will usually be just the combinatorial pattern of the triangles that is important. We drew the circular Farey diagram the way we did at the beginning of the chapter because it looks more symmetric and is easier to draw since one does not have to figure out the exact positions of the vertices.

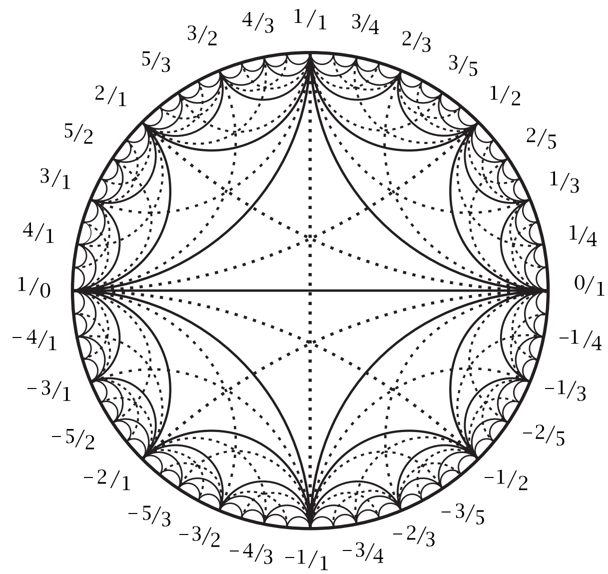
The next figure shows the relationship between the new circular Farey diagram and Pythagorean triples (a,b,c) using the formulas $(a,b,c) = (2pq, p^2-q^2, p^2+q^2)$ that we found in the previous chapter. The vertex with label p/q thus has coordinates $(x,y) = (a/c, b/c) = \left(2pq/p^2+q^2, p^2-q^2/p^2+q^2\right)$.



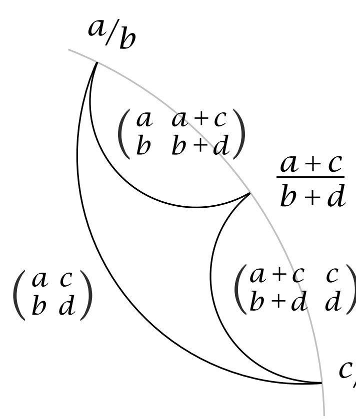
The construction we have described for the Farey diagram involves an inductive process where more and more edges and vertex labels are added in succession. With a construction like this it is not easy to tell by a simple calculation whether or not two given rational numbers a/b and c/d are joined by an edge in the diagram. Fortunately there is such a criterion:

Proposition 1.1. *For each pair of fractions a/b and c/d, including $\pm 1/0$, there exists an edge in the Farey diagram with endpoints labeled a/b and c/d if and only if the determinant $ad-bc$ of the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is equal to ± 1.*

What this means is that if one starts with the rational numbers together with $\pm 1/0$ arranged in order around a circle and one inserts circular arcs inside this circle meeting it perpendicularly and joining each pair of fractions a/b and c/d such that $ad - bc = \pm 1$, with the circular arc replaced by a diameter in case a/b and c/d are diametrically opposite on the circle, then no two of these arcs will cross, and they will divide the interior of the circle into nonoverlapping curvilinear triangles. This is really quite remarkable when you think about it, and it does not happen for other values of the determinant besides ± 1. For example, for determinant ± 2 the edges would be the dotted arcs in the figure at the right. Here there are three arcs crossing in each triangle of the original Farey diagram, and these arcs divide each triangle of the Farey diagram into six smaller triangles.



Proof: First we show by an inductive argument that for an edge in the diagram joining two fractions a/b and c/d the associated matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ has determinant ± 1. The induction starts with the edge joining $\pm 1/0$ to $0/1$ where the determinant condition obviously holds. All the other edges are added in stages, first the four edges creating the two biggest triangles, then the eight edges creating the next four triangles, and so on. Consider a triangle created at some stage by adding a new vertex labeled $a+c/b+d$ as the mediant of vertices a/b and c/d from an earlier stage, as in the figure at the right. We may assume by induction that $ad - bc = \pm 1$ for the long edge of the triangle which was added at an earlier stage. The determinant condition then holds also for the two shorter edges of the triangle since $a(b+d) - b(a+c) = ad - bc$ and $(a+c)d - (b+d)c = ad - bc$. Thus the determinant condition continues to hold after each stage of the construction of the diagram, so it holds for all edges.



Now we prove the converse, the statement that if $ad - bc = \pm 1$ then there is an edge in the diagram joining a/b and c/d. We may assume $b \geq 0$ and $d \geq 0$ by multiplying both numerator and denominator of either fraction by -1 if necessary, which multiplies the determinant $ad - bc$ by -1. The order of the two fractions a/b and c/d does not matter since interchanging the two columns of the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ also multiplies the determinant by -1. If b or d is 0, say $b = 0$, then the determinant condition becomes $ad = \pm 1$ so $d = 1$ and $a = \pm 1$. In this case the fractions a/b

and c/d are $\pm 1/0$ and $c/1$ so they lie at the ends of an edge of the diagram, one of the vertical edges to $1/0$ in the upper halfplane version of the diagram. Thus for the rest of the proof we may assume $b > 0$ and $d > 0$.

The previous figure shows that adding a new triangle to the diagram creates two new edges corresponding to matrices obtained from $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ by replacing one of the columns by the sum of the two columns. To finish the proof we will show that for each matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ of determinant ± 1 with $b > 0$ and $d > 0$ it is possible to perform a finite sequence of the inverse operations of subtracting one column from the other and end up with a matrix that we already know corresponds to an edge in the diagram. We will do this by always subtracting the column with smaller second entry from the column with larger second entry, so that these two entries remain positive. We stop the process when the two entries in the second row become equal. For example, here is how the process works for the matrix $\begin{pmatrix} 3 & 7 \\ 8 & 19 \end{pmatrix}$:

$$\begin{pmatrix} 3 & 7 \\ 8 & 19 \end{pmatrix} \to \begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Here the last matrix corresponds to the edge joining $1/1$ and $0/1$. Reversing the steps reducing $\begin{pmatrix} 3 & 7 \\ 8 & 19 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, we are adding one column to the other at each stage so each new matrix produced in this way corresponds to an edge of the diagram. In particular this shows that the original matrix $\begin{pmatrix} 3 & 7 \\ 8 & 19 \end{pmatrix}$ corresponds to an edge of the diagram.

For the general argument we start with a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ of determinant ± 1 with $b > 0$ and $d > 0$. If $b \neq d$ then we subtract the column with smaller second entry from the column with larger second entry, and repeat this operation until the two entries in the second row are equal. We cannot get a 0 in the second row since this would mean that the previous matrix already had equal entries in the second row. Once we get a matrix with equal entries in the second row, these entries will divide the determinant which is ± 1 so these entries must be 1. Thus the matrix is of the form $\begin{pmatrix} a & c \\ 1 & 1 \end{pmatrix}$, with determinant $a - c = \pm 1$ so a and c differ by 1. The corresponding fractions are then $n/1$ and $n+1/1$ for some integer n, and there is an edge of the diagram joining these two fractions, one of the large semicircles in the upper halfplane diagram. Hence when we reverse the sequence of column subtractions by performing a sequence of column additions, each successive matrix will correspond to an edge of the diagram and in particular $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ will correspond to an edge of the diagram. □

The sign of the determinant $ad - bc$ has a simple interpretation for fractions a/b and c/d with positive denominators since in this case the inequality $ad - bc > 0$ is equivalent to $a/b > c/d$ and $ad - bc < 0$ is equivalent to $a/b < c/d$. Thus the sign of the determinant tells which of a/b or c/d is larger.

Here is an interesting consequence of the preceding proposition:

Corollary 1.2. *The mediant rule for labeling the vertices in the Farey diagram always produces labels a/b that are fractions in lowest terms.*

This would follow automatically if it was always true that the mediant of two fractions in lowest terms is again in lowest terms, but this is not always the case. For example, the mediant of $1/3$ and $2/3$ is $3/6$, and the mediant of $2/7$ and $3/8$ is $5/15$. Somehow cases like this do not occur in the Farey diagram.

Before deducing the corollary let us introduce a bit of standard terminology. For a fraction a/b to be in lowest terms means that a and b have no common factor greater than 1. This is equivalent to saying that the prime factorizations of a and b have no prime factor in common. When this is the case we say that a and b are *coprime*. An alternative terminology is to say that a and b are *relatively prime*.

Proof: From the way the Farey diagram is constructed, each labeled vertex a/b is joined to some other labeled vertex c/d by an edge of the diagram. By the easier half of Proposition 1.1 we have $ad - bc = \pm 1$. This implies that a and b are coprime since any common divisor of a and b must divide the products ad and bc, hence also the difference $ad - bc = \pm 1$, but the only divisors of ± 1 are ± 1. □

Proposition 1.1 can also be used to prove another basic fact about the Farey diagram:

Proposition 1.3. *Every fraction p/q in lowest terms occurs as the label on some vertex in the Farey diagram.*

Proof: We may assume p and q are nonzero since $0/1$ and $1/0$ certainly occur as labels in the diagram. Since the negative labels in the diagram are just the negatives of the positive labels, we can assume p and q are in fact positive. It will suffice to show that if p and q are coprime, then there is an edge in the diagram whose endpoints are labeled p/q and r/s for some integers r and s. By Proposition 1.1 this is equivalent to the existence of integers r and s such that $ps - qr = \pm 1$.

Consider a matrix $\begin{pmatrix} x & y \\ p & q \end{pmatrix}$ where the integers x and y are yet to be determined. In the proof of Proposition 1.1 there was a procedure for repeatedly subtracting the column with smaller second entry from the column with larger second entry until a matrix with equal second entries is obtained. Subtracting one column from the other does not affect coprimeness of the two second entries, so when the procedure is applied to a matrix $\begin{pmatrix} x & y \\ p & q \end{pmatrix}$ with p and q coprime, the result is a matrix whose second entries are equal and coprime, so these entries must be 1. Now let us choose a matrix of determinant ± 1 whose lower two entries are 1, say the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. If we start with this matrix and apply the reverse of the sequence of operations performed on $\begin{pmatrix} x & y \\ p & q \end{pmatrix}$ to get 1's in the second row, the resulting sequence of operations of adding one column to the other converts $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ into a matrix $\begin{pmatrix} r & s \\ p & q \end{pmatrix}$ of the same determinant ± 1. This means that we have found integers r and s such that $rq - ps = \pm 1$, or equivalently $ps - qr = \pm 1$. □

Implicit in this proof is a method for solving Diophantine equations of the form $px - qy = \pm 1$ for any two given coprime positive integers p and q. In Section 2.3 we will make this procedure explicit and streamline it to be more efficient.

Exercises

1. There is another version of the Farey diagram in which the vertex labeled p/q is placed at the point (q,p) in the plane, so p/q is the slope of the line through the origin and (q,p). The edges of this new Farey diagram are straight line segments connecting the pairs of vertices that are connected in the original Farey diagram. For example there is a triangle with vertices $(1,0)$, $(0,1)$, and $(1,1)$ corresponding to the big triangle in the upper half of the circular Farey diagram. With this model of the Farey diagram the operation of forming the mediant of two fractions just corresponds to standard vector addition $(a,b) + (c,d) = (a+c, b+d)$.

What you are asked to do in this problem is just to draw the portion of the new Farey diagram consisting of all the triangles whose vertices (q,p) satisfy $0 \le q \le 5$ and $0 \le p \le 5$. Note that since fractions p/q labeling vertices are always in lowest terms, the points (q,p) such that q and p have a common divisor greater than 1 are not vertices of the diagram.

2. Consider a vertex of the Farey diagram labeled a/b with $b > 1$. Show that of all the labels on vertices connected to the a/b vertex by an edge of the diagram, exactly two have denominator smaller than b.

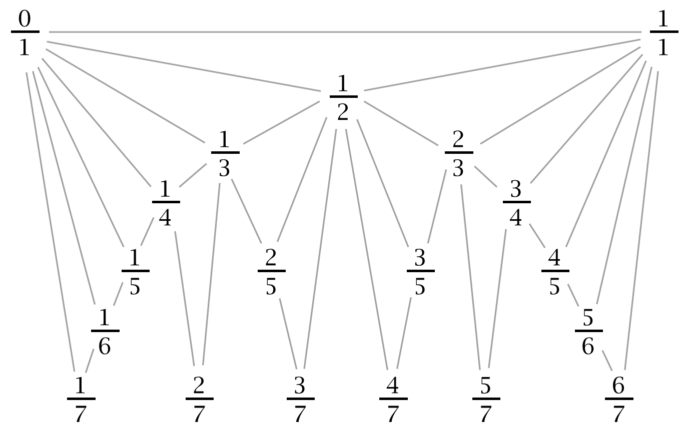
3. If a/b, c/d, and e/f are fractions in lowest terms such that e/f is the mediant of a/b and c/d, is it necessarily true that there is a triangle in the Farey diagram with vertices a/b, c/d, and e/f? Give either a proof or a counterexample.

4. (a) Reduce each of the matrices $\begin{pmatrix} 7 & 3 \\ 16 & 7 \end{pmatrix}$ and $\begin{pmatrix} 67 & 14 \\ 24 & 5 \end{pmatrix}$ to either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by repeatedly subtracting one column from the other as in the proof of Proposition 1.1.
(b) Use Proposition 1.1 to show that this can be done for any matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with non-negative entries and determinant ± 1.

5. Determine whether the following statement is always true: If $a/b < a'/b'$ and $c/d < c'/d'$ then the mediant of a/b and c/d is less than the mediant of a'/b' and c'/d'.

1.2 Farey Series

We can build the set of rational numbers by starting with the integers and then inserting in succession the halves, thirds, fourths, fifths, sixths, and so on. Let us look at what happens if we restrict to rational numbers between 0 and 1. Starting with 0 and 1 we first insert $1/2$, then $1/3$ and $2/3$, then $1/4$ and $3/4$, skipping $2/4$ which we already have, then inserting $1/5$, $2/5$, $3/5$, and $4/5$, then $1/6$ and $5/6$, etc. A natural way to depict this way of listing rational numbers between 0 and 1 is to place the terms with equal denominators in successive rows, with line segments connecting each new term to its two nearest neighbors among the terms in the previous rows as shown in the figure for denominators up to 7. Inspecting the figure, it appears that each new term is the mediant of its two neighbors, and we will show that in fact this always happens. This means that we are just constructing a straight-line version of the part of the Farey diagram between $0/1$ and $1/1$.



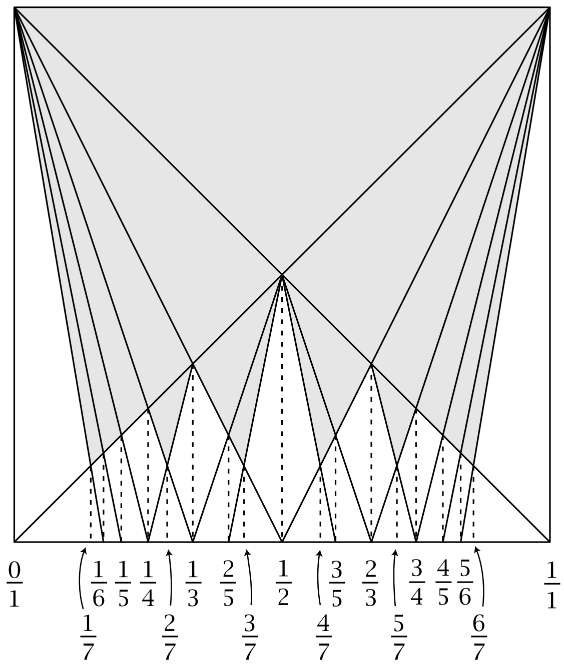
The discovery of this curious mediant property in the early 1800s was initially attributed to a geologist and amateur mathematician named Farey, although it turned out that he was not the first person to have noticed it. In spite of this confusion, the sequence of fractions a/b between 0 and 1 with denominator less than or equal to a given number n is called the nth *Farey series* F_n. For example, here is F_7:

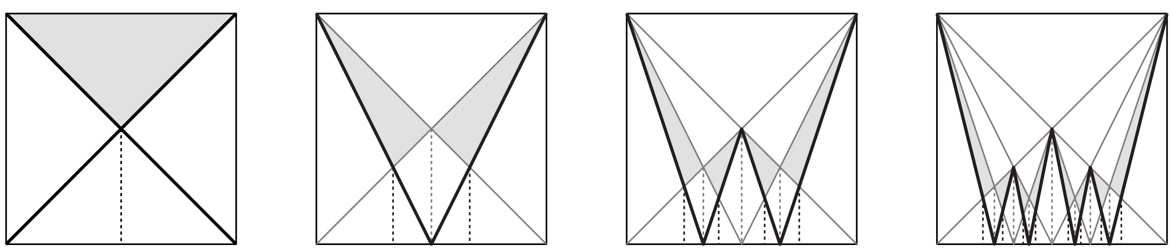
$$\frac{0}{1}\ \frac{1}{7}\ \frac{1}{6}\ \frac{1}{5}\ \frac{1}{4}\ \frac{2}{7}\ \frac{1}{3}\ \frac{2}{5}\ \frac{3}{7}\ \frac{1}{2}\ \frac{4}{7}\ \frac{3}{5}\ \frac{2}{3}\ \frac{5}{7}\ \frac{3}{4}\ \frac{4}{5}\ \frac{5}{6}\ \frac{6}{7}\ \frac{1}{1}$$

These numbers trace out the up-and-down path across the bottom of the preceding figure. For the next Farey series F_8 we would insert $1/8$ between $0/1$ and $1/7$, $3/8$ between $1/3$ and $2/5$, $5/8$ between $3/5$ and $2/3$, and finally $7/8$ between $6/7$ and $1/1$.

The mediant property of the Farey series F_n holds not just when each new term is added as described above, but in fact for every three consecutive terms of the series. For example in F_7, the mediant of $1/5$ and $2/7$ is $3/12 = 1/4$, so the mediant fraction must be reduced to lowest terms when the middle of the three denominators is not greater than the other two. This extended mediant property of Farey series will be deduced from a more general fact about mediants later in this section.

A more compact version of the preceding diagram that puts the part of the Farey diagram between 0 and 1 into a square is shown in the figure at the right. This can be constructed in stages as indicated in the sequence of figures below. Starting with a square, one first adds its diagonals and a vertical line from their intersection point down to the bottom edge of the square. The vertical line divides the region below the shaded triangle into two quadrilaterals. Each quadrilateral has one of its diagonals already present, and for the second stage of the construction we add the other diagonal and drop a vertical line from the intersection point of the two diagonals down to the bottom edge of the square. The process is then repeated for each subsequent step, adding a second diagonal in each unshaded quadrilateral and then a vertical line from the intersection point of the two diagonals down to the bottom edge of the square.



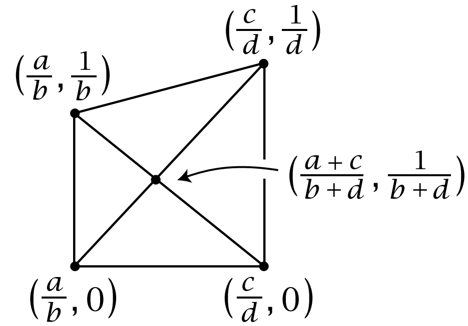


Let us choose the square to lie in in the upper halfplane with sides of length 1, with the bottom edge of the square along the x-axis and the lower left corner of the square at the origin. We then use the mediant rule to label the vertices of the shaded triangles as we proceed downward in the square, starting with the labels $0/1$ and $1/1$ at the upper left and right corners of the square. The positions of the vertices within the square can be described very simply:

- The vertex labeled a/b is located at the point $(a/b, 1/b)$.

This is obviously true for the vertices labeled $0/1$ and $1/1$ at the upper corners of the square, and also for the vertex labeled $1/2$ at the centerpoint $(1/2, 1/2)$ of the square. For the remaining vertices we proceed by induction downward in the diagram. Each step of the induction is a special case of the following geometric characterization of mediants:

- For any two fractions a/b and c/d consider a quadrilateral in the xy-plane with vertices at the points shown in the figure at the right. Then the diagonals of the quadrilateral intersect at $(a+c/b+d, 1/b+d)$. Thus the mediant of a/b and c/d is the x-coordinate of the intersection point of the diagonals.



To verify this let us first show that $(a+c/b+d, 1/b+d)$ is on the diagonal from $(a/b, 0)$ to $(c/d, 1/d)$. To do this it suffices to show that the line segments from $(a/b, 0)$ to $(a+c/b+d, 1/b+d)$ and from $(a+c/b+d, 1/b+d)$ to $(c/d, 1/d)$ have the same slope. These slopes are

$$\frac{1/b+d}{a+c/b+d - a/b} = \frac{b}{b(a+c) - a(b+d)} = \frac{b}{bc-ad}$$

and
$$\frac{1/d - 1/b+d}{c/d - a+c/b+d} = \frac{b+d-d}{c(b+d) - d(a+c)} = \frac{b}{bc-ad}$$

so they are equal. The same argument works for the other diagonal by interchanging a/b and c/d. Thus the diagonals intersect at the point $(a+c/b+d, 1/b+d)$.

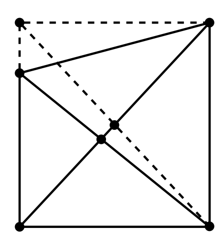
Note that the denominator $bc - ad$ in the slope formulas above is ± 1 when a/b and c/d are the endpoints of an edge of the Farey diagram. Thus each diagonal line in the square Farey diagram has integer slope, and this integer is, up to sign, the denominator of the rational number where the line meets the x-axis.

The fact that the y-coordinate of the vertex labeled a/b in the square diagram is $1/b$ implies that the successive Farey series can be obtained by taking the vertices that lie above the line $y = 1/2$, then the vertices above $y = 1/3$, then above $y = 1/4$, and so on. This explains why each new term inserted when F_n is enlarged to F_{n+1} is the mediant of its two neighbors in F_n. We see also that at most one new term of F_{n+1} is inserted between any two adjacent terms of F_n since there cannot be two triangles in the diagram with the same upper edge but different lower vertices.

From the geometric interpretation of mediants given above we can deduce a general fact about mediants:

- The mediant of two fractions a/b and c/d is closer to the fraction with larger denominator, unless the two denominators are equal in which case the mediant is halfway between the two fractions.

This can be seen by comparing the diagonals of the quadrilateral for a/b and c/d with the diagonals of the rectangle obtained by moving one of the upper two vertices of the quadrilateral vertically to the same height as the other upper vertex.



The following general fact justifies the earlier assertion that each term in the Farey series F_n is the mediant of the two adjacent terms.

- For rational numbers $a/b < c/d < e/f$, if there are edges in the Farey diagram joining a/b to c/d and c/d to e/f, then c/d is the mediant of a/b and e/f reduced to lowest terms.

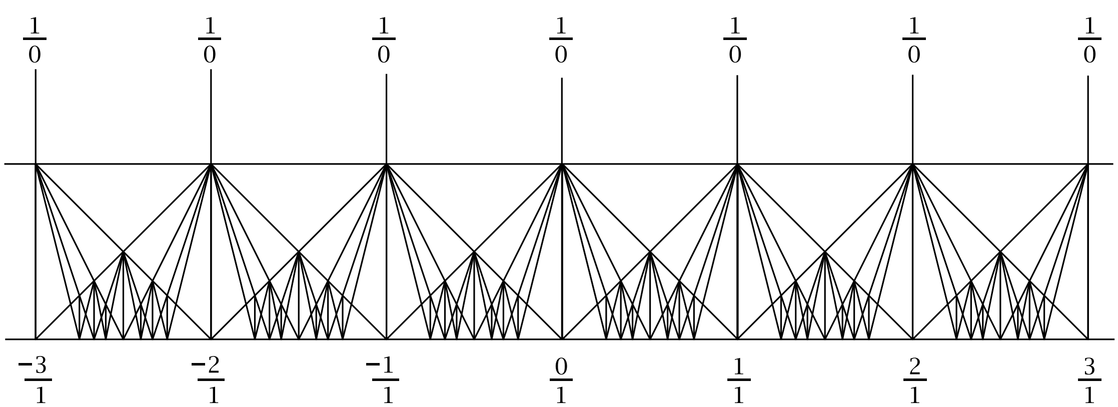
To see this we compute the mediant $a+e/b+f$. The assumption that there is an edge joining a/b and c/d means that $ad - bc = \pm 1$, so if $a/b < c/d$ we have $ad < bc$ so $ad - bc = -1$ and hence $bc - ad = 1$. Similarly, if there is an edge joining c/d to e/f we have $de - cf = 1$. From these equations we have:

$$a = ade - acf \qquad b = bde - bcf$$
$$e = bce - ade \qquad f = bcf - adf$$
$$\text{hence} \quad \frac{a+e}{b+f} = \frac{c(be-af)}{d(be-af)} = \frac{c}{d}$$

Note that in the last step, the fraction c/d is in lowest terms since we assumed c/d is a vertex of the Farey diagram, and the factor $be - af$ that we canceled to obtain c/d is, up to sign, the determinant of the matrix $\begin{pmatrix} a & e \\ b & f \end{pmatrix}$ associated to the pair of fractions a/b and e/f, and this determinant is ± 1 exactly when a/b and e/f are joined by an edge in the diagram.

As an example, all the fractions $m-1/m$ are connected to $1/1$ in the Farey diagram as are the fractions $n+1/n$ on the other side of $1/1$, and the mediant of $m-1/m$ and $n+1/n$ is $m+n/m+n = 1/1$. Here the number $be - af$ that is being canceled to get a fraction in lowest terms is $m(n+1) - (m-1)n = m + n$ which is the number of triangles in the Farey diagram between the edges from $1/1$ to $m-1/m$ and $n+1/n$.

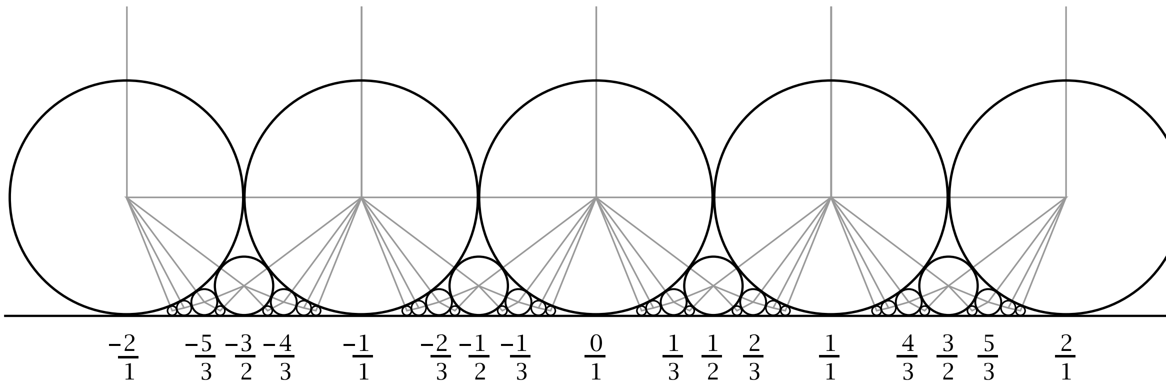
We can form a linear version of the full Farey diagram by placing copies of the square diagram we have been considering side by side along the x-axis:



Here the vertical segments in the horizontal strip of squares are not part of the resulting Farey diagram, which consists just of the triangles with nonvertical edges, along with the infinite "triangles" above the strip with a vertex at $1/0$. The original halfplane Farey diagram can be obtained from this linear Farey diagram by shrinking each vertical segment in the horizontal strip down to its lower endpoint while bending each straight edge of a triangle into a semicircle with endpoints on the x-axis.

Ford Circles

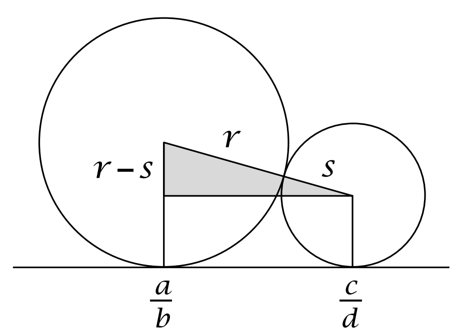
Another version of the Farey diagram can be constructed from an array of circles in the upper halfplane tangent to the x-axis and to each other as in the following figure:



This arrangement of tangent circles can be built in stages, starting with circles of diameter 1 tangent to the x-axis at the integer points. At the next stage a smaller circle is inserted in each gap between adjacent pairs of circles from the first stage. This creates new gaps, and one then puts a still smaller circle in each of these gaps. The process can then be repeated indefinitely all along the x-axis.

If we connect the centers of each pair of tangent circles by a line segment passing through the point of tangency, we obtain a pattern of triangles that is combinatorially equivalent to the pattern of triangles in the linear Farey diagram, but compressed closer to the x-axis. The vertices of these triangles are the centers of the various tangent circles, and we can label these centers by rational numbers, starting with an integer label $n/1$ at the center of the large circle tangent to the x-axis at the point n, and then labeling all the other centers by applying the mediant rule repeatedly.

The surprising thing about this construction is that the circle whose center is labeled a/b is tangent to the x-axis at exactly the point a/b on the x-axis. This can be verified as follows. For an edge of the Farey diagram with endpoints labeled a/b and c/d let us draw two circles tangent to each other and tangent to the x-axis at the points a/b and c/d. Let the radii of these two circles be r and s respectively. Note that r and s are not uniquely determined by a/b and c/d. In fact we can choose r arbitrarily and then this determines s, with s becoming small as r becomes large, and vice versa. We can find a formula for how r and s are related by applying the Pythagorean theorem to the right triangle



shown in the figure. The horizontal side of this triangle has length $|c/d - a/b|$ and the vertical side has length $|r - s|$. The condition for the two circles to be tangent is that the hypotenuse of the triangle has length $r + s$. Thus we have:

$$(r-s)^2 + \left(\frac{c}{d} - \frac{a}{b}\right)^2 = (r+s)^2$$

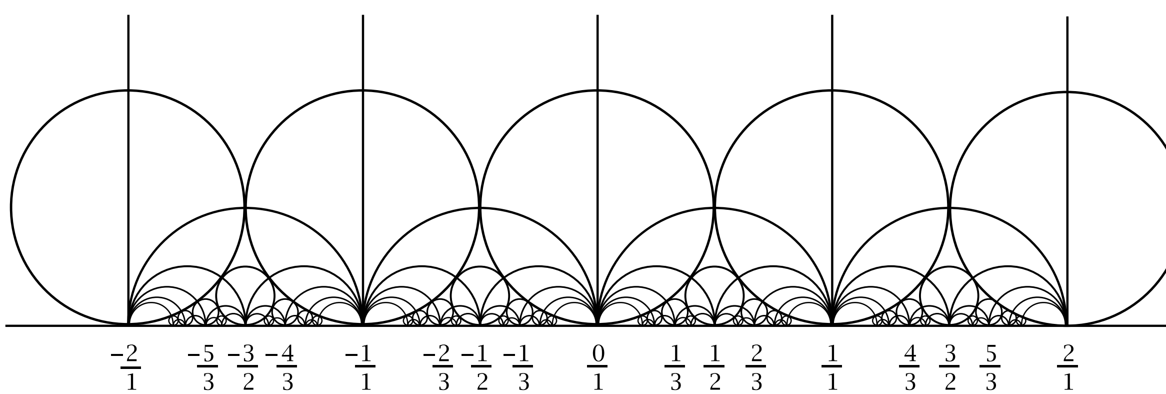
This simplifies to:

$$\left(\frac{bc - ad}{bd}\right)^2 = 4rs$$

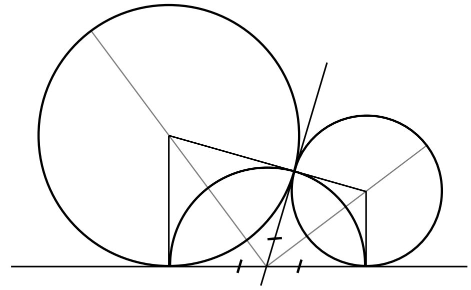
Since we assumed the fractions a/b and c/d were the endpoints of an edge in the Farey diagram, we have $ad - bc = \pm 1$ so the preceding equation simplifies further to $(\frac{1}{bd})^2 = 4rs$. The easiest way to assure that this holds is to let $r = 1/2b^2$ and $s = 1/2d^2$, so that r depends only on a/b and s depends only on c/d. Thus we are choosing the diameter of each circle to be the reciprocal of the square of the denominator of the fraction where the circle is tangent to the x-axis. This is consistent with how we chose the initial large circles tangent to the x-axis at integer points. Then when we build the Farey diagram inductively by adding more and more vertices labeled according to the mediant rule, each new vertex labeled $a+c/b+d$ between vertices labeled a/b and c/d is the center of a circle of diameter $1/(b+d)^2$ tangent to the x-axis at $a+c/b+d$ and tangent to each of the two circles labeled a/b and c/d of diameters $1/b^2$ and $1/d^2$ that are tangent to the x-axis at a/b and c/d.

The circles tangent to the x-axis constructed in this way are called *Ford circles* after their discoverer L. R. Ford. From the formula for their diameters we see that the Ford circles whose diameter is greater than a fixed number are just the ones associated to the fractions in a Farey series, if we restrict attention to the circles tangent to the x-axis at points between 0 and 1.

Another very nice feature of Ford circles is that when we superimpose them on the upper halfplane Farey diagram, the semicircles of the Farey diagram intersect the Ford circles orthogonally at the points of tangency of the Ford circles:



The fact that the circles and semicircles intersect orthogonally at the tangency points of the circles can be verified by considering the tangent lines to the circles at the points where two circles are tangent. The key fact is that for any two nonparallel tangent lines to a circle, the distances from the points of tangency to the intersection point of the two tangent lines are equal. This is because reflecting across the radial line through the intersection point takes one tangent line to the other.



Exercises

1. Compute the Farey series F_{10}.

2. Draw a figure showing how Ford circles are positioned in a circular Farey diagram by the following procedure. Start with a circle C of radius 1 which will be the outer boundary of the Farey diagram. Next, draw two tangent circles of radius $1/2$ inside C and tangent to C at two opposite points of C. Label these two tangency points $1/0$ and $0/1$. Now continue drawing smaller circles inside C with the same tangency patterns as the Ford circles in the upper halfplane Farey diagram, and label the tangency points of these circles with C according to the mediant rule. After a number of these circles have been drawn, superimpose the semicircles of the Farey diagram itself.

3. Suppose two Ford circles tangent to the x-axis at points a/b and c/d are tangent to each other. Show that the point of tangency between the two circles is the point

$$\left(\frac{ab+cd}{b^2+d^2}, \frac{1}{b^2+d^2}\right)$$

so in particular the coordinates of this point are rational. *Hint*: What proportion of the way along the line segment joining the two centers is the point of tangency? This same proportion will apply to x-coordinates and y-coordinates separately.