In 1969, probably on the evening of Nov. 22, following Adrien's Bourbaki lecture on the work of Frisch and Guenot, I had a conversation with him that has influenced the rest of my life.

Walking back to the Luxembourg station, I complained that I didn't understand the Kodaira-Spencer cohomology classes that came up in deformation theory.

He invited me to a drink in the Café du Luxembourg, and the explanation he gave is somewhere in the background of everything I have done since.

I did not realize it at the time, but during this lecture he touched on most of the research he performed in the period 1960-1974.

I will try to reproduce what he told me, as a device to tie together the main works of this period.

So how does cohomology appear in deformation theory?

We will describe this “informally”

first in terms of almost-complex structures
related to Dolbeault cohomology

and then in terms of deforming changes of coordinates
related to Čech cohomology.

The Dolbeault approach

Let X be a complex manifold.

Let $\Phi(X)$ be the space of almost-complex structures on X, and $\Phi^{int}(X)$ be the subset of integrable almost-complex structures.

The deformation space Def(X) of X "is" the quotient of $\Phi^{int}(X)$ by the action of $\mathrm{Diff}(X)$ acting by pullback.

Trouble with this construction

$$X = \left(\mathbb{C}^2 - \begin{bmatrix}0\\0\end{bmatrix}\right) \Big/ \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}^{\mathbb{Z}}.$$

Then $\mathrm{Def}(X)$ is not Hausdorff. The manifolds

$$X_t = \left(\mathbb{C}^2 - \begin{bmatrix}0\\0\end{bmatrix}\right) \Big/ \begin{bmatrix}2 & t\\0 & 2\end{bmatrix}^{\mathbb{Z}}$$

are all diffeomorphic to X, and all isomorphic when $t \neq 0$, so correspond to a single point P of $\mathrm{Def}(X)$.

But X is not isomorphic to X_t when $t \neq 0$. The point $P \in \mathrm{Def}(X)$ is not the base point of $\mathrm{Def}(X)$ corresponding to X itself, but contains the "base point" in its closure.

For now we ignore the difficulties involved in giving $\Phi^{int}(X)$ a reasonable structure, or making sense of the quotient, and proceed as if the above made sense.

Since X is a complex manifold, it corresponds to a "base" almost complex structure $\phi_0 \in \Phi(X)$, and a neighborhood of ϕ_0 can be identified with a neighborhood U of 0 in $A^{0,1}_X(TX)$, as follows.

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$$u_\alpha(\xi) = \xi + \alpha(\xi)$$

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People used to quasiconformal mappings might think of α as a Beltrami form.

then α corresponds to the almost-complex structure where multiplication by i is given by

$$i \bullet_\alpha \xi = u_\alpha^{-1}(iu_\alpha(\xi))$$

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This is well defined on the (big) open set

$$U \subset A_X^{0,1}(TX)$$

where u_α is an isomorphism, and ϕ_0 corresponds to $\alpha = 0$.

In this chart, the condition for integrability is written

$$F : \alpha \mapsto \overline{\partial}\alpha - [\alpha \wedge \alpha]$$

The map

$$F : A_X^{0,1}(TX) \to A_X^{0,2}(TX)$$

is an "analytic" map with derivative $\overline{\partial}$.

The space $U \cap \Phi^{int}(X)$ is defined by the equation

$$U \cap \Phi^{int}(X) := F^{-1}(0).$$

Now recall that we want to quotient $\Phi^{int}(X)$ by the action of $\mathrm{Diff}(X)$ acting by pullback.

Again pretend that there are no difficulties (non-Hausdorff quotients, etc) in taking this quotient. Then a subspace transverse to the orbit of ϕ_0 should represent the quotient.

Let $G : \mathrm{Diff}(X) \to \Phi(X)$ be the inclusion of this orbit, i.e.,

$$G(f) = f^* \phi_0.$$

The derivative of G at the identity is the map

$$DG(id) : A_X^{0,0}(TX) \to A_X^{0,1}(TX)$$

given by

$$DG(id)(\xi) = -\overline{\partial}\xi.$$

Summary

The deformation space Def(X) is the quotient of the space $\Phi^{int}(X)$ of integrable almost complex structures on X by the action of $\mathrm{Diff}(X)$.

This leads to a sequence of spaces

$$\mathrm{Diff}(X) \xrightarrow{G} \Phi(X) \xrightarrow{F} A_X^{0,2}(TX).$$

The corresponding sequence of derivatives is

$$A_X^{0,0}(TX) \xrightarrow{-\bar{\partial}} A_X^{0,1}(TX) \xrightarrow{\bar{\partial}} A_X^{0,2}(TX),$$

i.e., the beginning of the Dolbeault resolution of TX.

Thus, Def(X) "should" be a space with Zariski tangent space $H^1(X, TX)$, and defined in $H^1(X, TX)$ by an equation

$$\overline{F} : H^1(X, TX) \to H^2(X, TX)$$

with leading term the quadratic cup-bracket

$$\alpha \mapsto [\alpha \wedge \alpha].$$

The Čech approach

A smooth proper family X_t of manifolds parametrized by a manifold T is a manifold $\mathbf{X}$ together with a map $p : \mathbf{X} \to T$ such that for every $x \in \mathbf{X}$, there exists a neighborhood $T' \subset T$ of $t := p(x)$, a neighborhood U of x in $p^{-1}(t)$ and an isomorphism $f : T' \times U \to \mathbf{X}$ to its image such that the diagram

$$\begin{array}{ccc} T' \times U & \xrightarrow{f} & \mathbf{X} \\ \downarrow & & \downarrow p \\ T' & \hookrightarrow & T \end{array}$$

commutes.

Suppose $t_0 \in T$ is a basepoint, such that $X = p^{-1}(t_0)$, that $\mathcal{U} = \{U_i\}$ is a finite cover by open sets as above, and that $T' \subset T$ is sufficiently small that $f_i : T' \times U_i \to \mathbf{X}$ is defined for all i. Then the maps

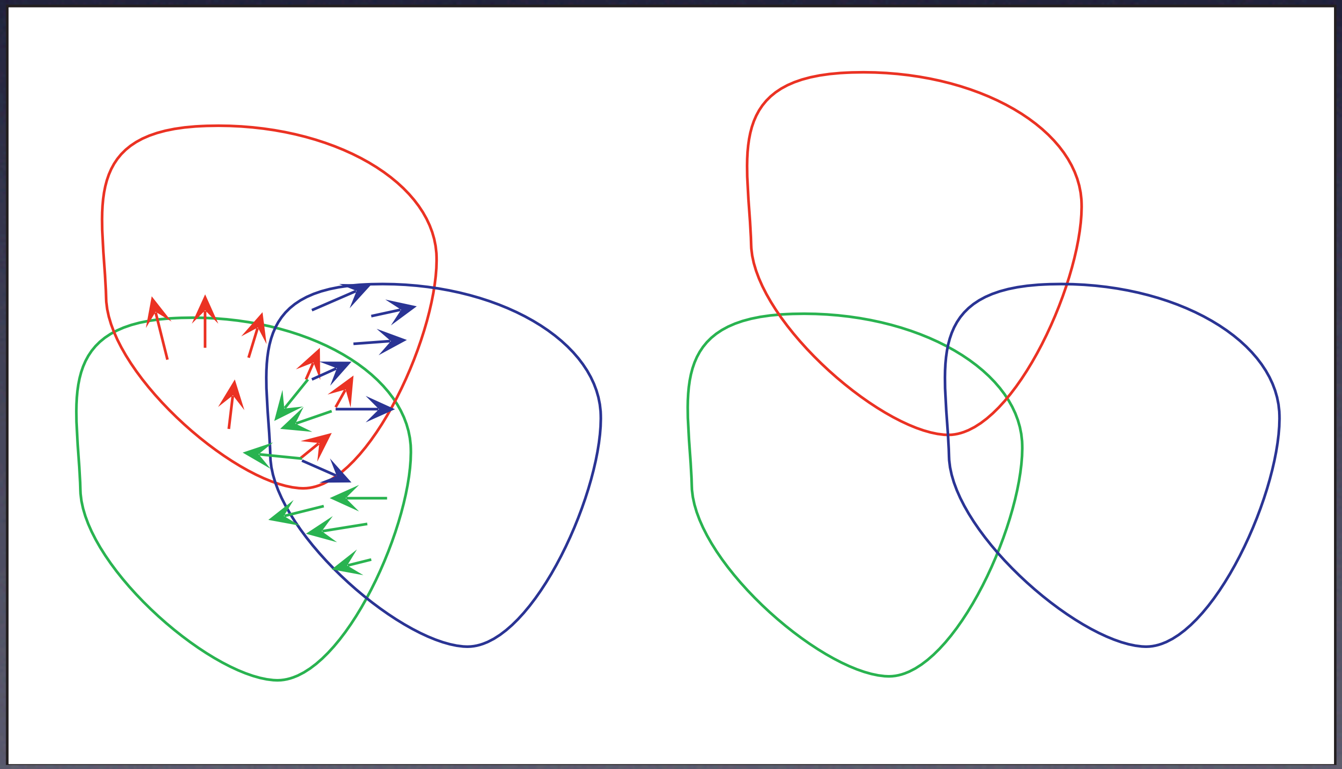
$$f_{i,j}(t) = f_j(t) \circ f_i^{-1}(t)$$

provide parametrized "change of coordinate maps".

The domain of definition of $f_{i,j}(t)$ includes any compact subset of $U_i \cap U_j$ for t sufficiently close to t_0. Thus we can consider the derivative

$$\xi_{i,j} = \frac{d}{dt} f_{i,j}(t)|_{t=t_0}$$

which is a vector field on $U_i \cap U_j$.



Differentiating the relation
$f_{i,k}(t) = f_{j,k}(t) \circ f_{i,j}(t)$ leads to the relation

$$\xi_{i,k} = \xi_{j,k} + \xi_{i,j}$$

i.e., the $\xi_{i,j}$ form a Čech 1-cocycle for the cover $\mathcal{U}$, with values in the sheaf T_X of holomorphic vector-fields on X.

Note that the relation $f_{i,k}(t) = f_{j,k}(t) \circ f_{i,j}(t)$ contains further information, about the brackets of the $\xi_{i,j}$ in particular.

It is not hard to show that this construction leads to the *Kodaira-Spencer* map

$$T_{t_0}T \to H^1(X, TX)$$

which measures "how fast" the manifolds X_t are deforming.

Roughly the description so far is the content of that conversation at the Café du Luxembourg. He had a clear idea of what to do next...

This makes it very tempting to try to define Def(X) as a quotient of the space of gluing maps

$$f_{i,j} : U_i \cap U_j \to U_i \cap U_j$$

by an appropriate equivalence relation. Adrien eventually succeeded in doing this, and even in doing it when X has singularities, but that only happened in 1974, and it is time to describe some of the steps along the way.

The Cartan Seminar 1960-1961

The first works of Adrien on deformations of complex spaces are apparently the first 4 talks in the Cartan seminar 1960-1961.

In these lectures, Adrien (then 25) shows that he has a remarkable command of Kodaira-Spencer theory.

He investigates various specific deformation spaces, in particular complex tori, bundles of complex tori, and Hopf surfaces.

The main theme of these lectures is finding obstructed classes in $H^1(X,TX)$

In the simplest case, this means finding classes $\alpha \in H^1(X, TX)$ such that $[\alpha \wedge \alpha] \neq 0$.

But there are higher obstructions. Sometimes the quadratic term of the equation $F : H^1(X, TX) \to H^2(X, TX)$ defining $\mathrm{Def(X)} \subset \mathrm{H^1(X, TX)}$ vanishes without F vanishing. This leads to higher obstructions; as far as I know these are the only examples of such things in the literature.

Kuranishi's theorem

In December 1964, in the Bourbaki seminar, Adrien gave a new proof (7 pages) of Kuranishi's theorem (41 pages).

Recall the informal definition

$$\mathrm{Def}(X) = \Phi^{int}(X)/\mathrm{Diff}(X).$$

Kuranishi managed to make something like this mathematically correct.

To prove that anything is a manifold, one needs the implicit function theorem, which requires *Banach* manifolds so that the tangent spaces are Banach spaces, and also that derivatives have closed complemented images.

Denote by ${}^{n+r}A_X^{0,p}(TX)$ the space of $(0,p)$ forms with values in TX that are n times differentiable with nth derivatives *Hölder* of exponent r for some $0 < r < 1$. It turns out that the sequence

$${}^{n+1+r}A_X^{0,0}(TX) \overset{-\overline{\partial}}{\to} {}^{n+r}A_X^{0,1}(TX) \overset{\overline{\partial}}{\to} {}^{n-1+r}A_X^{0,2}(TX),$$

is a sequence of Banach spaces that computes $H^1(X, TX)$, and the differentials do have closed complemented images.

Thus we are tempted to construct

$$ {}^{n+r}\Phi^{int}(X) := {}^{n+r}F^{-1}(0) $$

where

$$ {}^{n+r}F : {}^{n+r}A_X^{0,1}(TX) \xrightarrow{\overline{\partial}} {}^{n-1+r}A_X^{0,2}(TX) $$

is still given by $F(\alpha) = \overline{\partial}\alpha - [\alpha \wedge \alpha]$, and to define

$$ \mathrm{Def}(X) = {}^{n+r}\Phi^{int}(X)/\, {}^{n+1+r}\mathrm{Diff}(X). $$

The first part works: we can define

$$ {}^{n+r}\Phi^{int}(X) := {}^{n+r}F^{-1}(0). $$

The result is a *Banach analytic space*, usually not a manifold.

Taking the quotient does not work: usually the quotient is not Hausdorff, and Def(X) does not exist.

What does work is locally intersecting ${}^{n+r}\Phi^{int}(X)$ with a subspace transverse to the orbit of ϕ_0 under ${}^{n+1+r}\mathrm{Diff}(X)$. This constructs a finite-dimensional analytic space which is a *local versal deformation* of X, not universal since whole strata of the space can correspond to isomorphic manifolds. This happens for Hopf surfaces, as we saw.

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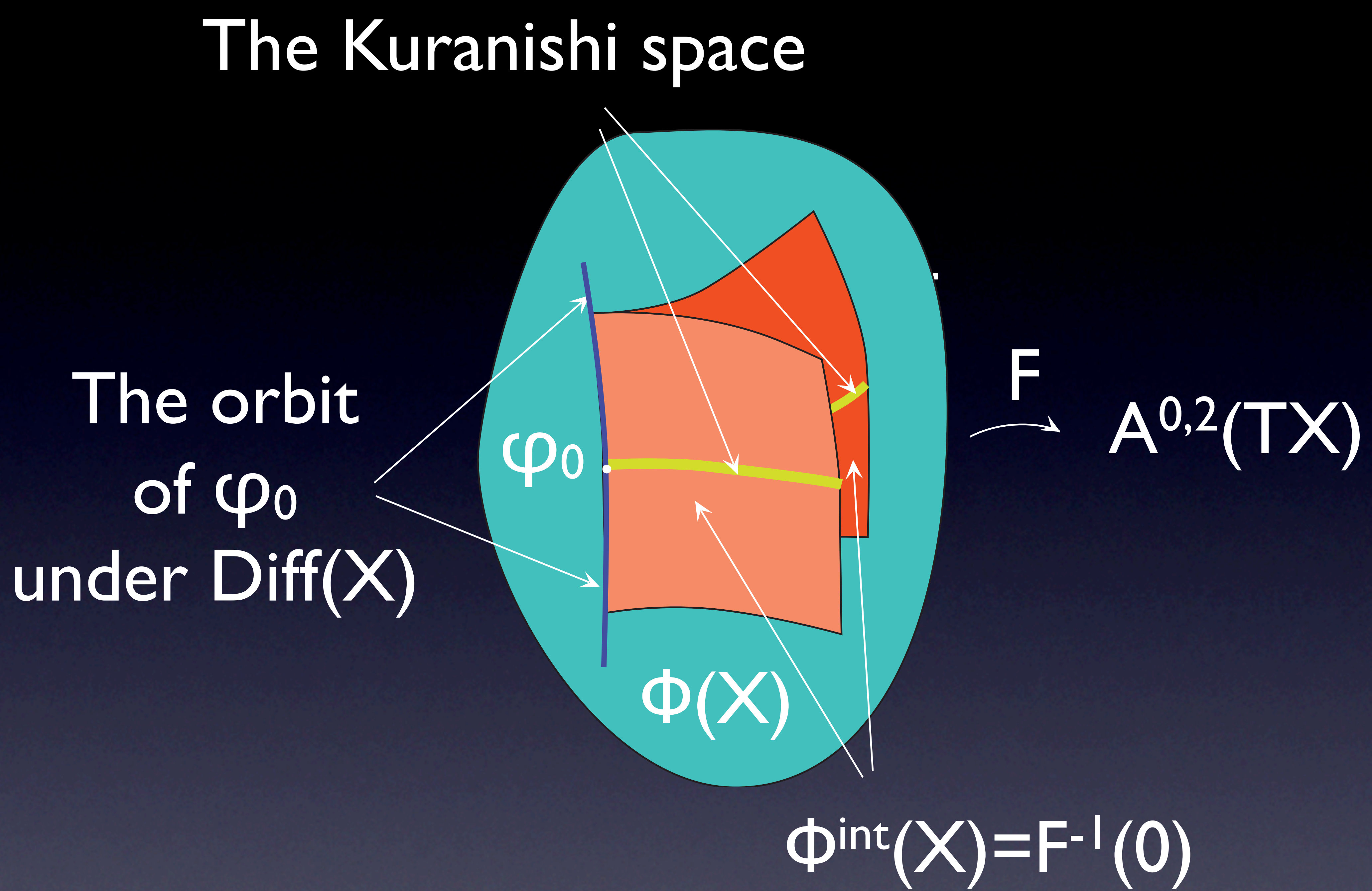
The picture on the next slide illustrates the construction but of course all the objects are supposed to be infinite dmensional



Note that whole strata of $\Phi^{int}(X)$
may consist of isomorphic manifolds.
this happens already for Hopf surfaces.

Proving all this is quite delicate; Adrien's lemma 1 illustrates the power of *privileged neighborhoods* (to be defined soon).

The Thesis

Adrien's thesis constructs an analog of the
Hilbert scheme
in analytic geometry.

Let X be a complex analytic space.
Adrien constructs a space $H(X)$ of all compact analytic subspaces of X.

He shows that this space represents the functor which associates to an analytic space S the set of proper flat families of subspaces of X parametrized by S

and he shows that this space is locally finite dimensional.

To accomplish this, Adrien had to create
an immense arsenal of tools.

Because he is dealing with analytic spaces,
not manifolds,
almost-complex structures and the Dolbeault
theory do not work, and he has to use Čech techniques.

Adam Epstein will speak in much greater detail
about the thesis, so here I will only give a few pointers.

Privileged neighborhoods put the problem in the setting of Banach spaces and Banach algebras.

Locally the space is defined by an ideal in a Banach algebra, which can be deformed, leading to a space of ideals which is a subset of the Grassmannian of closed complemented subspaces of the underlying Banach space.

The flatness and privilege principle says that the space of subspaces parametrized by this space of ideals is flat.

Finally, the local finite dimensionality follows from a non-linear Riesz perturbation theorem, related to the Cartan-Serre theorem.

Outline of the construction

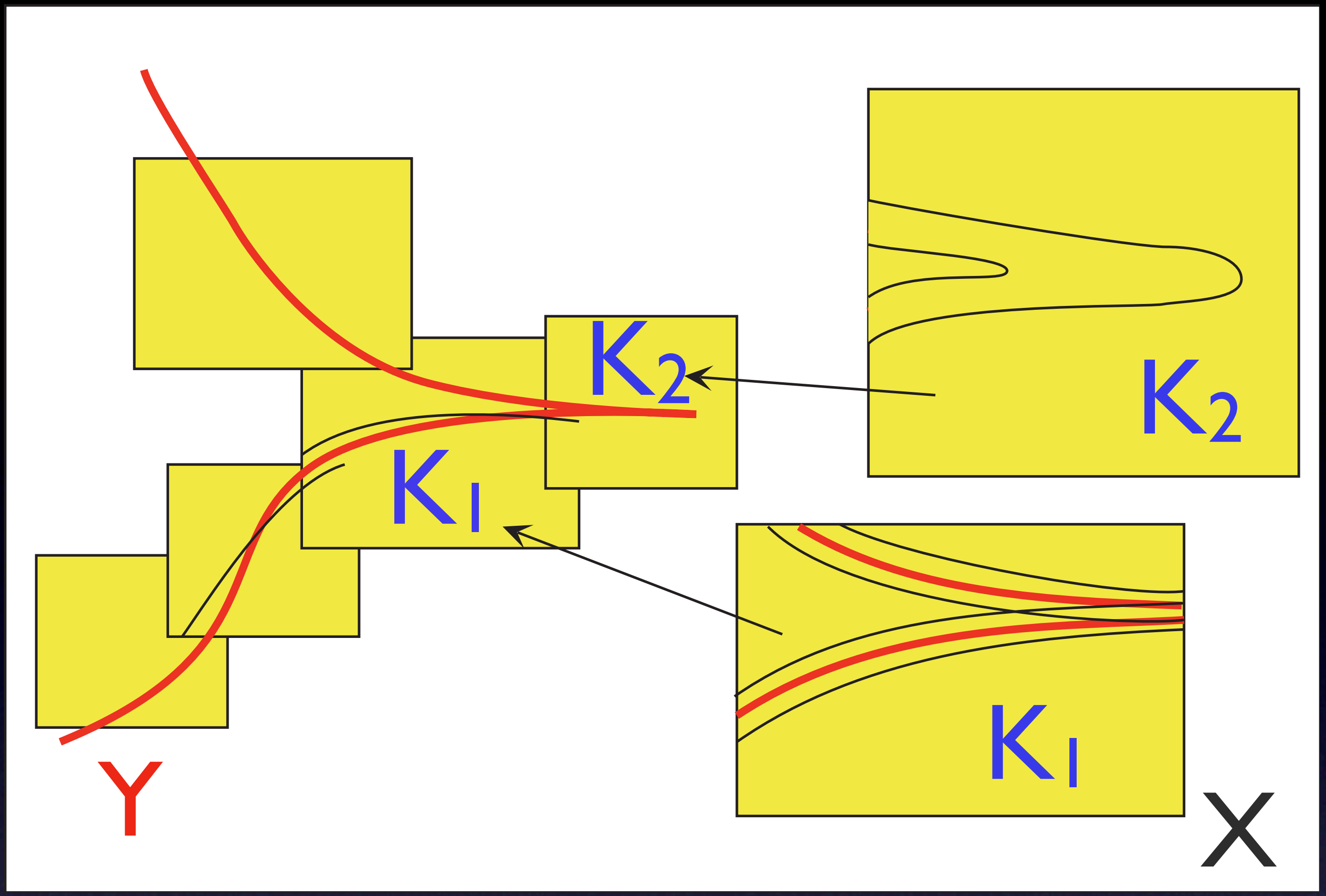
If K is a compact polydisk, denote by $B(K)$ the Banach algebra of continuous functions on K, analytic in the interior.

Let X be a complex analytic space, and $Y \subset X$ a compact subset. Cover Y by compact polydisks $K_1, \ldots, K_n$. In a neighborhood of K_i, Y is defined in X by functions $f_1, \ldots, f_m$. The restrictions $f|_{K_i}$ of these functions are in $B(K_i)$, hence generate an ideal in the Banach algebra $B(K_i)$.

The polydisk $K \subset X$ is *privileged* for $\mathcal{O}_Y$ if the corresponding ideal is a closed complemented subspace of $B(K)$ (as a Banach space).

The space $I(A)$ of closed complemented ideals in a Banach algebra A is a Banach analytic subspace of the Grassmann manifold of closed complemented subspaces of A.

When $A = B(K)$, this space parametrizes a family of subspaces of K that are *flat* over $I(B(K))$.



Deform $Y \cap K_i$, constructing a family of subspaces parametrized by

$$\prod_i I(B(K_i)).$$

It is not too hard to show that the subset $\Theta \subset \prod_i I(B(K_i))$ corresponding to subspaces that coincide on all $K_i \cap K_j$ is a Banach-analytic subspace.

But Θ is *not* locally a finite-dimensional analytic space. It is set-theoretically right, but its structure sheaf is wrong. The trickiest part of the construction is to reduce Θ so that it becomes the finite-dimensional space $H(X)$.

The moduli space for complex analytic spaces

In the summer of 1973, Adrien finally solved the local moduli problem for complex analytic spaces.

I was personally involved in this work: we worked out the broad outline during a canoe trip with Régine on the Mediterranean, from St Raphael to Nice.

Although we cosigned the announcement in the CRAS, I was definitely the very junior author.

The statement is the same as for Kuranishi's theorem.

Let X be a compact analytic space. There exist

(1) analytic spaces $\mathbf{X}$, T,

(2) a flat proper mapping $F : \mathbf{X} \to T$,

(3) a base point $t_0 \in T$, and an isomorphism $\phi : G^{-1}(t_0) \to X$,

which are versal in the sense that

For any proper flat mapping $G : \mathbf{Y} \to S$ with an isomorphism $\psi : G^{-1}(s_0) \to X$, there exists a neighborhood $S' \subset S$ of s_0, a mapping $f : S' \to T$ and an isomorphism $\mathbf{Y}_{S'} \to f^*\mathbf{X}$.

All the techniques needed for the thesis are used

in a harder context

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Privileged neighborhoods

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Flatness and privilege

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The space of ideals as a subspace of the Grassmanian

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The construction of a space Θ that is set-theoretically right

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in a harder context

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The construction of a space Θ that is set-theoretically right

The thinning-down construction which replaces this space by a finite-dimensional space.

Teichmüller Theory

Before continuing to the second main topic of Adrien's mathematics, holomorphic dynamics, I want to talk about the period 1974-1980, and more particularly Teichmüller theory and Strebel forms.

For one thing it ties in with the earlier work.

For another, Teichmüller theory has important consequences in holomorphic dynamics.

For a third, I was directly involved.

Teichmüller theory is the place where the deformation theory described above works best. Let X be a compact Riemann surface.

First $\Phi^{int}(X) = \Phi(X)$. Recall that the equations defining $\Phi^{int}(X)$ in $\Phi(X)$ take their values in $A^{0,2}_X(TX)$, and $A^{0,2}_X(TX) = 0$ this space vanishes when X is a Riemann surface. *All almost-complex structures are integrable on a Riemann surface.*

Next, the quotient $\Phi(X)/\mathrm{Diff}(X)$ is a Hausdorff space when X is a Riemann surface. This space $\mathrm{Def}(X)$ carries the structure of a complex analytic space, but is not smooth. Experience has shown that the best space to study is Teichmüller space $\mathcal{T}_X$. Let $\mathrm{Diff}^0(X)$ be the group of diffeomorphisms isotopic to the identity. Then

$$\mathcal{T}_X = \Phi(X)/\mathrm{Diff}^0(X).$$

The space $\mathcal{T}_X$ parametrizes a family of *marked Riemann surfaces*: there is a complex manifold Ξ_X and a smooth proper mapping $\pi : \Xi_X \to \mathcal{T}_X$ such that $\pi^{-1}(t)$ is the Riemann surface corresponding to t.

The theorem of my thesis is that π admits no analytic sections, except in genus 2 where it admits exactly 6 sections.

The proof consists mainly of a detailed study of the Teichmüller metric on Teichmüller space.

In the process of writing this thesis, and in particular showing that Grothendieck's Teichmüller space (described in the lectures 5-15 of the 1960-1961 Cartan seminar, to which Adrien contributed lectures 1-4) was the same as the Ahlfors-Bers Teichmüller space. I am only too well aware of how much Adrien helped when I wrote the thesis. Some parts are more his than mine.

In any case, he ended up an expert in Teichmüller theory and quasiconformal mappings, which turned out (eight years later) to be of great importance in holomorphic dynamics.

Another paper of 1975 solved a conjecture of Reich and Strebel. In it we proved that in the space of quadratic differentials on any Riemann surface of genus ≥ 2, those with closed horizontal trajectories are dense.

This is the beginning of a long development, still very much an active subject of research. Under the name of translation surfaces and rational billiards, there have been a great many contributions to the subject. I just attended the Ahlfors-Bers colloquium in Rutgers, and for something like half the contributions, that paper was in the direct ancestry of the results, even if the authors did not know it.

Holomorphic dynamics

It was already hard to summarize Adrien's work on moduli problems, spanning 14 years.

Summarizing the work in holomorphic dynamics is much harder yet, since it spans 27 years.

Still, it is quite easy to pinpoint the beginnings.

In 1978, I taught DEUG B (second-year calculus) at Orsay, and tried to introduce a bit of numerical mathematics, using what was available at the time: programmable calculators.

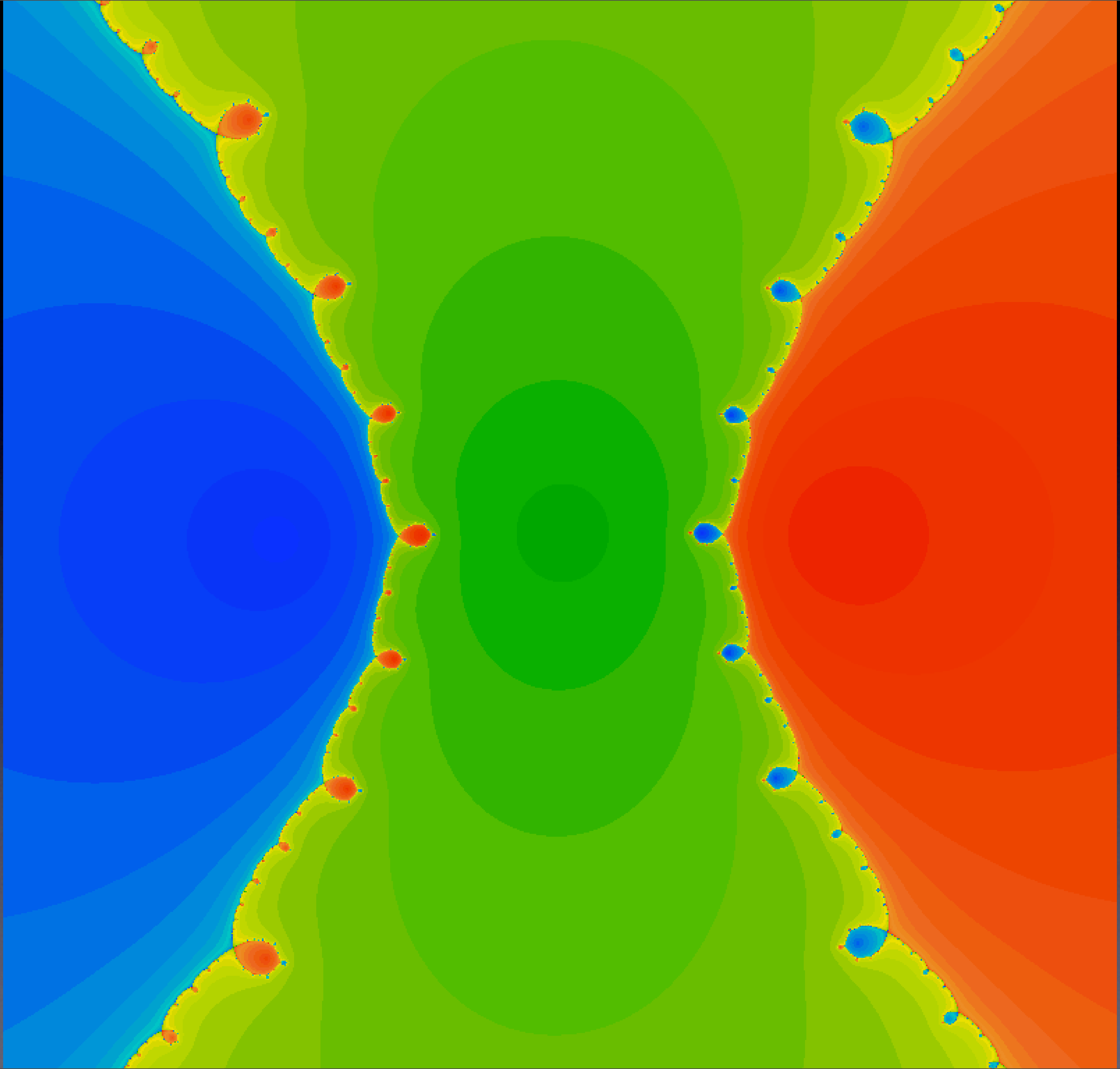
I had the students program Newton's method to solve cubic equations.

This was just possible on those antique machines:
32 program steps and 52 memories.

I assumed at the time that the experts
"knew where to start"
but it didn't take long to find out that the global behavior of Newton's method was a complete mystery.

This led us to use the (horrible) mini6 at Orsay to color the basins of the roots for Newton's method, leading to pictures like the following.

Actually, no color! The line printer put '+' and '|' .

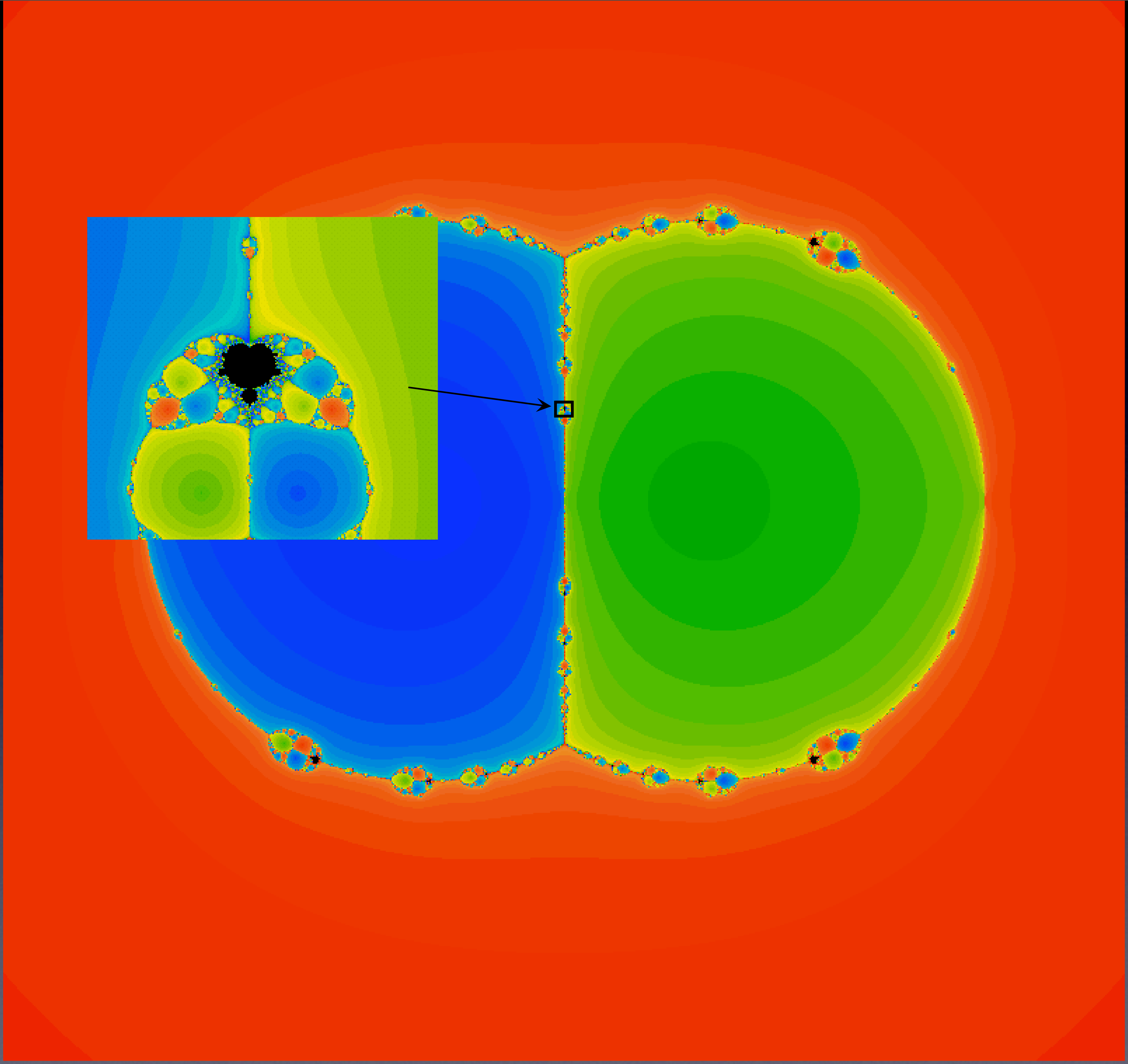


Sullivan was at IHES that year. He was aware of the work of Fatou and Julia,
in particular that
for a rational function, every attracting cycle attracts a critical point.

Since Newton's method N_p for a cubic polynomial p has three fixed critical points
at the roots, and one more critical point c_p, we saw how to produce parameter space pictures:

color p according to which root c_p is attracted to.

This leads to the following picture

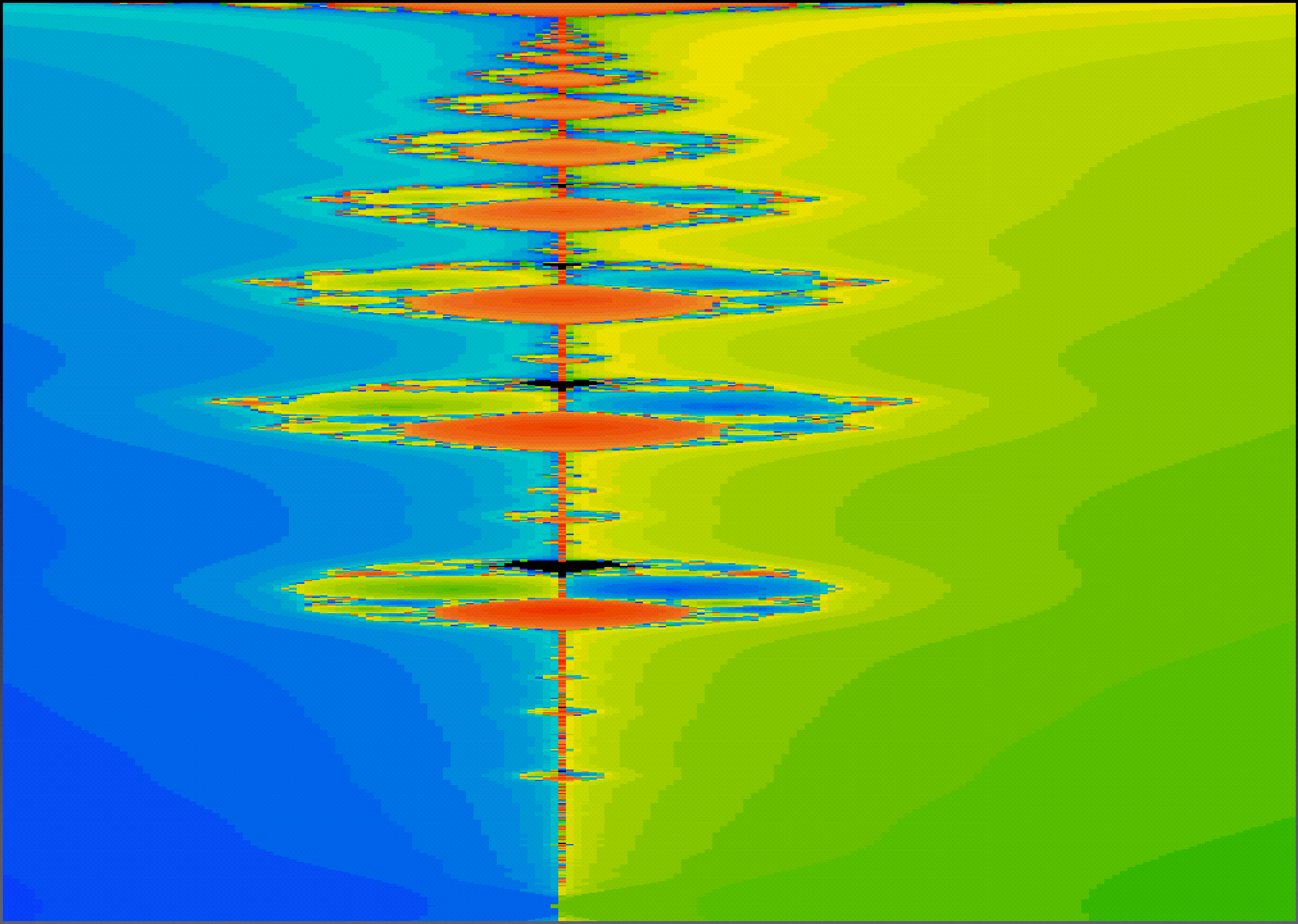


This should have lead to discovering the Mandelbrot set!

Unfortunately, I made a blow-up which didn't respect the aspect ratio.

We worked at very low resolution, printing on a line printer with 40 lines of 80 characters.

At this resolution, the picture looked like a mess.



During the next 3 years, Adrien poked at these pictures from various points of view.

He actually discovered many interesting things:

If p is a polynomial of degree $d \geq 2$, define the filled-in Julia set $K_p =$

$\{z \in \mathbb{C} \mid \text{the sequence } z, p(z), p(p(z)), \ldots \text{ is bounded.}$

The set K_p is connected if and only if all critical points of p belong to K_p.

He also found that if K_p is connected, there is a conformal isomorphism

$$\phi_p : \mathbb{C} - K_p \to \mathbb{C} - \overline{\mathbb{D}},$$

such that $\phi_p(p(z)) = p(z^d)$.

Further, he showed that if all critical points of p are attracted to attracting cycles, then ϕ_p^{-1} extends continuously to $\partial\mathbb{D}$.

This used the contracting properties of the Poincaré metric, something that has remained a central tool in the subject.

The year 1981-1982

This sets the stage for the academic year 1981-82.

It was a wonderfully fruitful year, with new results dropping on an almost daily basis.

I was back in France that year, quite intent on the pursuit of holomorphic dynamics.

The Mandelbrot set M is connected

Let $p_c(z) = z^2 + c$, and define

$$M = \{c \mid K_{p_c} \text{ is connected}\}.$$

The first big result of that year is

The Mandelbrot set M is connected

Let $p_c(z) = z^2 + c$, and define

$$M = \{c \mid K_{p_c} \text{ is connected}\}.$$

The first big result of that year is

The set M is connected.

This is really a problem about moduli spaces.
In particular, Adrien was particuliarly well placed for its study.

In all moduli space problems one tries to put on a moduli space whatever structure it classifies.

There is always something self-referential about moduli problems, and the connectivity of M is no exception.

The first step of the proof is to say that there always exists a map

$$\phi_c : (\mathbb{C}, \infty) \to (\mathbb{C}, \infty)$$

defined near ∞, such that $\phi_c(p_c(z)) = (\phi_c(z))^2$.

The second step is to say that if $c \notin M$, then c is in the domain of ϕ_c. So (this is the self-referential part) we can define

$$\Phi : \mathbb{C} - M \to \mathbb{C} - \overline{\mathbb{D}} \quad \text{by} \quad \Phi(c) = \phi_c(c).$$

The third step is to show that Φ is analytic, proper, of degree 1.

Thus we show that M is connected by constructing the conformal mapping of the complement.

Moreover, the mapping Φ has a dynamical meaning, leading to one of the key conjectures in the field:

MLC,

which can be stated in two ways

M is locally connected,

or

The map

$$\Phi^{-1} : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - M$$

extends to the boundary $\partial\mathbb{D}$.

A second key result of 1981-82 was the discovery of matings.

Let c_1, c_2 be two points of M, and $\gamma_k : \mathbb{R}/\mathbb{Z} \to K_{c_k}$ be defined by $\gamma_k(t) = \phi_{c_k}^{-1}(e^{2\pi i t})$.

Set

$$X_{c_1,c_2} = K_{c_1} \sqcup K_{c_2} / \sim$$

where the equivalence relation is given by setting $\gamma_1(t) \sim \gamma_2(-t)$

The polynomials p_{c_1} and p_{c_2} together define a mapping

$$f_{c_1,c_2} : X_{c_1,c_2} \to X_{c_1,c_2}.$$

It is far from obvious that X_{c_1,c_2} is ever homeomorphic to a sphere, but often it is. If so, and if f_{c_1,c_2} is conjugate to a rational function, then f_{c_1,c_2} is called the *mating* of p_{c_1} and p_{c_2}.

Adrien conjectured that any two polynomials that do not belong to conjugate limbs of the Mandelbrot set can be mated.

This is the analog of the *double limit theorem* in the theory of Kleinian groups. It was proved for post-critically finite polynomials by Tan Lei.

Quasiconformal mappings

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function. The Fatou set Ω_f is the set of $\mathbf{z} \in \overline{\mathbb{C}}$ which have a neighborhood on which the family of iterates $f^{\circ n}$ is normal.

It is easy to show that $f(\Omega_f) = f^{-1}(\Omega_f) = \Omega_f$. In particular, the components of Ω_f themselves form a dynamical system. Sullivan proved that there are no wandering components of Ω_f:

Every component of Ω_f is eventually periodic.

The problem was asked by Fatou: it was at least 60 years old when it was solved.

The technique was at least as important as the solution: it introduced quasiconformal techniques in holomorphic dynamics.

Adrien and I immediately saw how useful this technique could be; within a week we had proved the

Polynomial-like theorem.

Let $U' \subset U$ be simply-connected Riemann surfaces, with U' relatively compact in U, and $f : U' \to U$ be a proper analytic map of degree $d \geq 2$. Define

$$K_f = \bigcap_n f^{-n}(U).$$

Such a mapping is called a polynomial-like mapping; the standard example is to take f a polynomial, U a big disk and $U' = f^{-1}(U)$.

The straightening theorem for polynomial-like mappings says that:

For every polynomial-like mapping $f : U' \to U$ of degree d, there exists a polynomial p of degree d, a neighborhood V of K_p, and a quasiconformal homeomorphism $\phi : U \to V$ such that $\phi \circ f = p \circ \phi$ on U'. Moreover, we can choose ϕ so that $\bar{\partial}\phi = 0$ on K_f. Moreover, if K_f is connected, then p is unique up to conjugacy by an affine mapping.

This theorem has been a central result in all the work on renormalization...

Sullivan, Henri Epstein, McMullen, Yoccoz, Shishikura, Lyubich, Yampolsky

From 1982 on, I can’t hope to give the history.

The main thing I regret is not talking about
parabolic implosion.

This concerns the discontinuous changes in the dynamics of polynomials and rational functions when they acquire parabolic cycles.

It is parallel to geometric limits in the theory of Kleinian groups

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He is sorely missed