Some new approaches To Hénon mappings

Some new approaches To Hénon mappings

Low Dimensional mappings
Stony Brook, June 8-13, 2009

Some new approaches To Hénon mappings

Low Dimensional mappings
Stony Brook, June 8-13, 2009

John Hubbard,
with
Chris Lipa, Remus Radu and Reluca Tanase

In joint work with Koch, Lipa, Radu, Tanase
I have investigated Hénon mappings,
more particularly

In joint work with Koch, Lipa, Radu, Tanase
I have investigated Hénon mappings,
more particularly

- 4-dimensional topological models

In joint work with Koch, Lipa, Radu, Tanase
I have investigated Hénon mappings,
more particularly

- 4-dimensional topological models
- Pinched ball models

In joint work with Koch, Lipa, Radu, Tanase
I have investigated Hénon mappings,
more particularly

- 4-dimensional topological models
- Pinched ball models
- Analytic models

In joint work with Koch, Lipa, Radu, Tanase
I have investigated Hénon mappings,
more particularly

- 4-dimensional topological models
- Pinched ball models
- Analytic models
- Monodromy of horseshoes

In joint work with Koch, Lipa, Radu, Tanase
I have investigated Hénon mappings,
more particularly

- 4-dimensional topological models
- Pinched ball models
- Analytic models
- Monodromy of horseshoes
- Non-transversality locus

A Hénon mapping is written

$$H_{p,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - by \\ x \end{pmatrix}$$

A Hénon mapping is written

$$H_{p,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - by \\ x \end{pmatrix}$$

for some polynomial p of degree $d \geq 2$.

A Hénon mapping is written

$$H_{p,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - by \\ x \end{pmatrix}$$

for some polynomial p of degree $d \geq 2$.

When the Jacobian b satisfies $b = 0$, the Hénon mapping $H_{p,0}$ reduces to the polynomial $x \mapsto p(x)$.

A Hénon mapping is written

$$H_{p,b}\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} p(x) - by \\ x \end{array}\right)$$

for some polynomial p of degree $d \geq 2$.

When the Jacobian b satisfies $b = 0$, the Hénon mapping $H_{p,0}$ reduces to the polynomial

$$x \mapsto p(x).$$

If $b \neq 0$, the mapping $H_{p,b}$ is invertible:

$$H_{p,b}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{1}{b}(p(y) - x) \end{pmatrix}$$

The important dynamical objects are:

The important dynamical objects are:

$$K^{\pm}_{p,b} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \left\| H^{\circ n}_{p,b} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

The important dynamical objects are:

$$K^{\pm}_{p,b} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \left\| H^{\circ n}_{p,b} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

$$J^{\pm}_{p,b} = \partial K^{\pm}_{p,b}$$

The important dynamical objects are:

$$K^{\pm}_{p,b} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \left\| H^{\circ n}_{p,b} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

$$J^{\pm}_{p,b} = \partial K^{\pm}_{p,b}$$

$$K_{p,b} = K^{+}_{p,b} \bigcap K^{-}_{p,b}, \quad J_{p,b} = J^{+}_{p,b} \bigcap J^{-}_{p,b}$$

The important dynamical objects are:

$$K^{\pm}_{p,b} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \left\| H^{\circ n}_{p,b} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

$$J^{\pm}_{p,b} = \partial K^{\pm}_{p,b}$$

$$K_{p,b} = K^{+}_{p,b} \bigcap K^{-}_{p,b}, \quad J_{p,b} = J^{+}_{p,b} \bigcap J^{-}_{p,b}$$

What is the topology of these sets?

The important dynamical objects are:

$$K^{\pm}_{p,b} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \left\| H^{\circ n}_{p,b} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

$$J^{\pm}_{p,b} = \partial K^{\pm}_{p,b}$$

$$K_{p,b} = K^{+}_{p,b} \bigcap K^{-}_{p,b}, \quad J_{p,b} = J^{+}_{p,b} \bigcap J^{-}_{p,b}$$

What is the topology of these sets?

Are they connected?

The important dynamical objects are:

$$K_{p,b}^{\pm} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \left\| H_{p,b}^{\circ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

$$J_{p,b}^{\pm} = \partial K_{p,b}^{\pm}$$

$$K_{p,b} = K_{p,b}^{+} \bigcap K_{p,b}^{-}, \quad J_{p,b} = J_{p,b}^{+} \bigcap J_{p,b}^{-}$$

What is the topology of these sets?
Are they connected?
Do they carry invariant measures? ergodic measures?

The important dynamical objects are:

$$K^{\pm}_{p,b} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \left\| H^{\circ n}_{p,b} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

$$J^{\pm}_{p,b} = \partial K^{\pm}_{p,b}$$

$$K_{p,b} = K^{+}_{p,b} \bigcap K^{-}_{p,b}, \quad J_{p,b} = J^{+}_{p,b} \bigcap J^{-}_{p,b}$$

What is the topology of these sets?

Are they connected?

Do they carry invariant measures? ergodic measures?

What is the entropy?

Set

There are some cases where the answers are more or less known: small perturbations of hyperbolic polynomials. Suppose p is hyperbolic, and define

Set

There are some cases where the answers are more or less known: small perturbations of hyperbolic polynomials. Suppose p is hyperbolic, and define

$$\widehat{\mathbb{C}}_p = \varprojlim(\mathbb{C}, p) = \{(\ldots, z_{-2}, z_{-1}, z_0) \mid p(z_i) = z_{i+1}\}$$

Set

There are some cases where the answers are more or less known: small perturbations of hyperbolic polynomials. Suppose p is hyperbolic, and define

$$\widehat{\mathbb{C}}_p = \varprojlim(\mathbb{C}, p) = \{(\ldots, z_{-2}, z_{-1}, z_0) \mid p(z_i) = z_{i+1}\}$$

Choose $\mathbb{D}$ a disk so large that $J_p \subset \mathbb{D}$, and define $f_p : J_p \times \mathbb{D} \to J_p \times \mathbb{D}$ by

$$f_p(\zeta, z) = \left(p(\zeta), \zeta + \varepsilon \frac{z}{p'(\zeta)} \right)$$

Set

There are some cases where the answers are more or less known: small perturbations of hyperbolic polynomials. Suppose p is hyperbolic, and define

$$\widehat{\mathbb{C}}_p = \varprojlim(\mathbb{C}, p) = \{(\dots, z_{-2}, z_{-1}, z_0) \mid p(z_i) = z_{i+1}\}$$

Choose $\mathbb{D}$ a disk so large that $J_p \subset \mathbb{D}$, and define $f_p : J_p \times \mathbb{D} \to J_p \times \mathbb{D}$ by

$$f_p(\zeta, z) = \left(p(\zeta), \zeta + \varepsilon \frac{z}{p'(\zeta)} \right)$$

where $\varepsilon > 0$ is sufficiently small that f_p is defined and injective. Set

There are some cases where the answers are more or less known: small perturbations of hyperbolic polynomials. Suppose p is hyperbolic, and define

$$\widehat{\mathbb{C}}_p = \varprojlim(\mathbb{C}, p) = \{(\ldots, z_{-2}, z_{-1}, z_0) \mid p(z_i) = z_{i+1}\}$$

Choose $\mathbb{D}$ a disk so large that $J_p \subset \mathbb{D}$, and define $f_p : J_p \times \mathbb{D} \to J_p \times \mathbb{D}$ by

$$f_p(\zeta, z) = \left(p(\zeta), \zeta + \varepsilon \frac{z}{p'(\zeta)}\right)$$

where $\varepsilon > 0$ is sufficiently small that f_p is defined and injective. Set

$$\check{\mathbb{C}}_p = \varinjlim(\mathbb{C}, f_p)$$

The projection $\widehat{\mathbb{C}}_p \to \mathbb{C}$ given by $(\ldots, z_{-2}, z_{-1}, z_0) \mapsto z_0$ makes $\widehat{\mathbb{C}}_p$ into a "ramified cover" whose fibers are Cantor sets, ramified above the post-critical set.

The projection $\widehat{\mathbb{C}}_p \to \mathbb{C}$ given by $(\ldots, z_{-2}, z_{-1}, z_0) \mapsto z_0$ makes $\widehat{\mathbb{C}}_p$ into a "ramified cover" whose fibers are Cantor sets, ramified above the post-critical set.

In the case where $p(z) = z^d$,

The projection $\widehat{\mathbb{C}}_p \to \mathbb{C}$ given by $(\ldots, z_{-2}, z_{-1}, z_0) \mapsto z_0$ makes $\widehat{\mathbb{C}}_p$ into a "ramified cover" whose fibers are Cantor sets, ramified above the post-critical set.

In the case where $p(z) = z^d$,

$\widehat{\mathbb{C}}_p$

The projection $\widehat{\mathbb{C}}_p \to \mathbb{C}$ given by $(\ldots, z_{-2}, z_{-1}, z_0) \mapsto z_0$ makes $\widehat{\mathbb{C}}_p$ into a "ramified cover" whose fibers are Cantor sets, ramified above the post-critical set.

In the case where $p(z) = z^d$, $\widehat{\mathbb{C}}_p$ is the infinite cone over the d-adic solenoid.

The projection $\widehat{\mathbb{C}}_p \to \mathbb{C}$ given by $(\ldots, z_{-2}, z_{-1}, z_0) \mapsto z_0$ makes $\widehat{\mathbb{C}}_p$ into a "ramified cover" whose fibers are Cantor sets, ramified above the post-critical set.

In the case where $p(z) = z^d$, $\widehat{\mathbb{C}}_p$ is the infinite cone over the d-adic solenoid.

In all cases, $\widehat{\mathbb{C}}_p$ has non-algebraic singularities anywhere the postcritical set accumulates, in particular attracting cycles.

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

$$\varinjlim(X, f)$$

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

$$\varinjlim(X, f)$$

is the quotient of $X \times \mathbb{N}$ by the equivalence

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

$$\varinjlim(X, f)$$

is the quotient of $X \times \mathbb{N}$ by the equivalence relation

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

$$\varinjlim (X, f)$$

is the quotient of $X \times \mathbb{N}$ by the equivalence relation

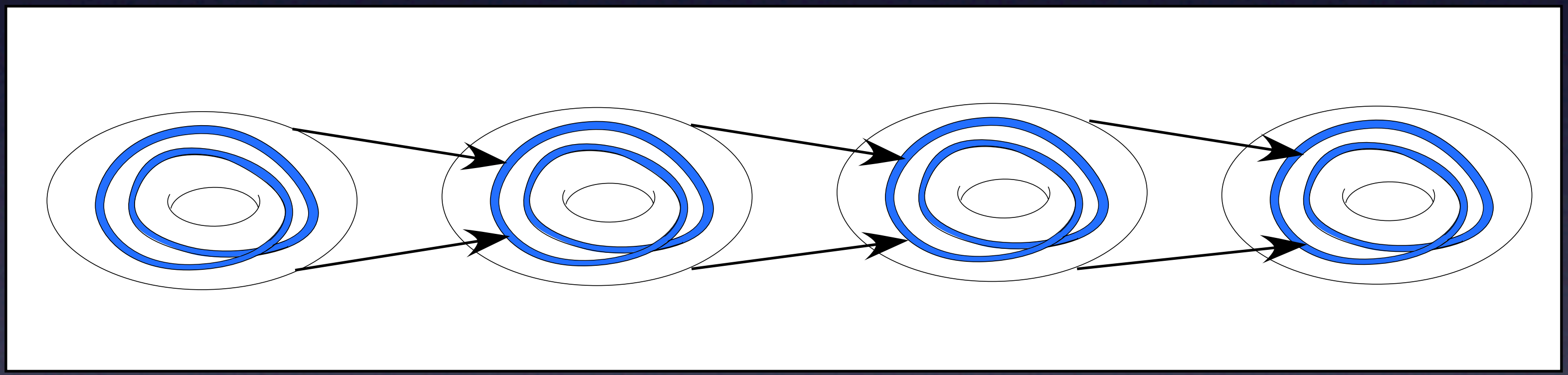
$$(x, n) \sim (f(x), n + 1)$$

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

$$\varinjlim(X, f)$$

is the quotient of $X \times \mathbb{N}$ by the equivalence relation

$$(x, n) \sim (f(x), n+1)$$

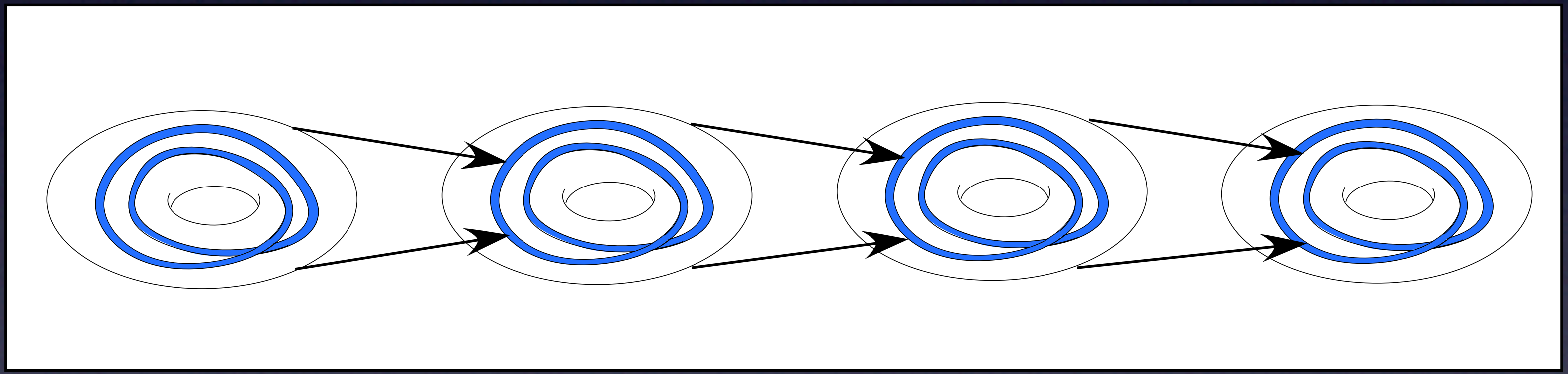


The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

$$\varinjlim(X, f)$$

is the quotient of $X \times \mathbb{N}$ by the equivalence relation

$$(x, n) \sim (f(x), n+1)$$



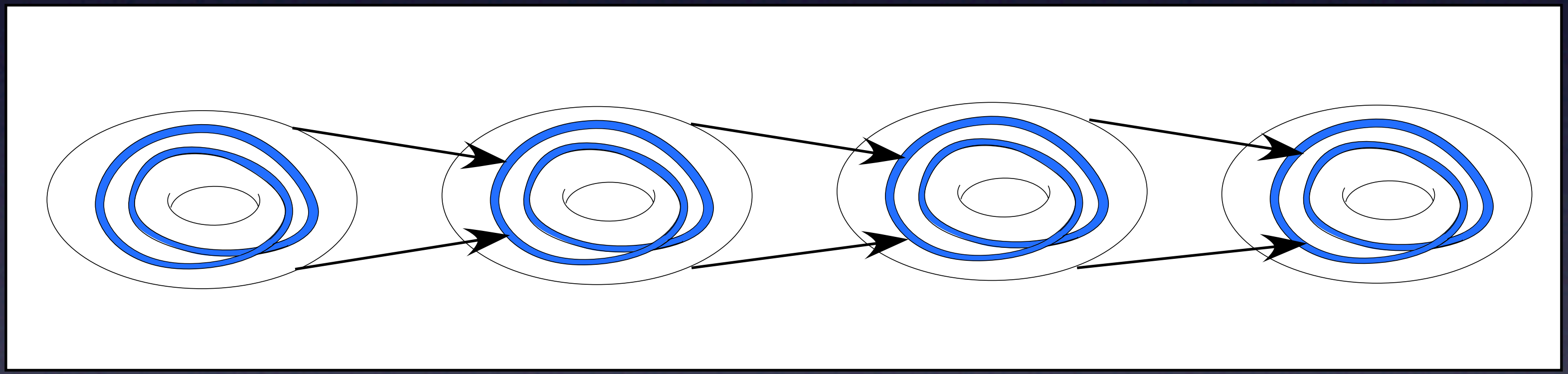
In this case, $p(z) = z^d$

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f : X \to X$ is a map, then

$$\varinjlim(X, f)$$

is the quotient of $X \times \mathbb{N}$ by the equivalence relation

$$(x, n) \sim (f(x), n+1)$$



In this case, $p(z) = z^d$

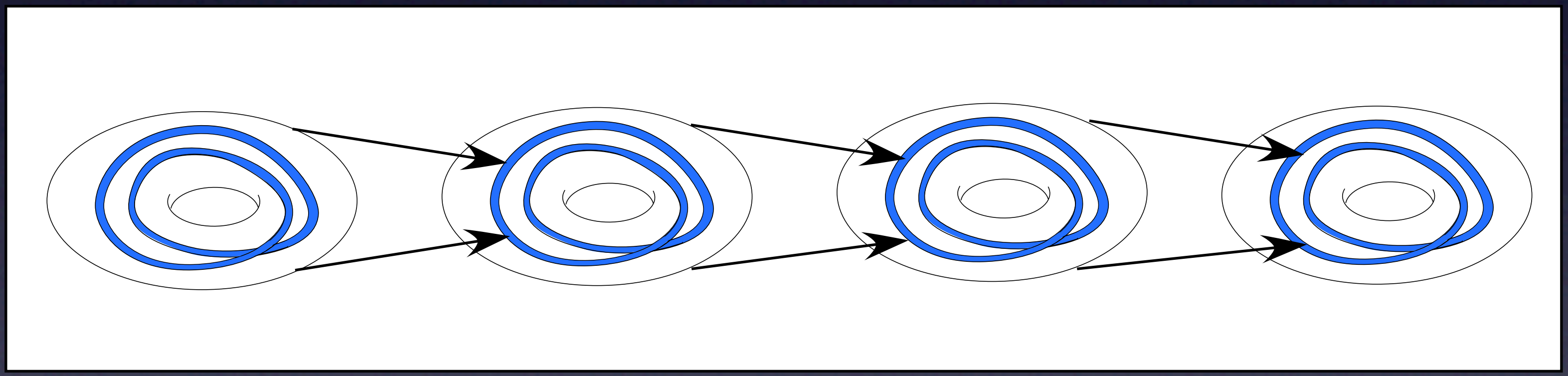
$$f_p : S^1 \times \mathbb{D} \to S^1 \times \mathbb{D}, \quad (\zeta, z) \mapsto (\zeta^d, \zeta + \varepsilon \frac{z}{d\zeta^{d-1}})$$

The space $\check{\mathbb{C}}_p$ is more exotic. Recall that if $f: X \to X$ is a map, then

$$\varinjlim(X, f)$$

is the quotient of $X \times \mathbb{N}$ by the equivalence relation

$$(x, n) \sim (f(x), n+1)$$



In this case, $p(z) = z^d$

$$f_p : S^1 \times \mathbb{D} \to S^1 \times \mathbb{D}, \quad (\zeta, z) \mapsto (\zeta^d, \zeta + \varepsilon \frac{z}{d\zeta^{d-1}})$$

the inductive limit is a 3-sphere with a d-adic solenoid removed.

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p$$

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p$ is given by

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p \qquad \text{is given by}$$

$$\hat{p}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, p(z_{-2}), p(z_{-1}), p(z_0))$$

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p$ is given by

$$\hat{p}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, p(z_{-2}), p(z_{-1}), p(z_0))$$

The inverse erases the last entry

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p$ is given by

$$\hat{p}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, p(z_{-2}), p(z_{-1}), p(z_0))$$

The inverse erases the last entry

$\check{p} : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p$

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p$ is given by

$$\hat{p}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, p(z_{-2}), p(z_{-1}), p(z_0))$$

The inverse erases the last entry

$\check{p} : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p$ is given by

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p$ is given by

$$\hat{p}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, p(z_{-2}), p(z_{-1}), p(z_0))$$

The inverse erases the last entry

$\check{p} : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p$ is given by

$$\hat{p}(x, n) = (f(x), n) = (x, n+1)$$

Both of the space $\widehat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ come with natural bijective dynamics:

$\hat{p} : \widehat{\mathbb{C}}_p \to \widehat{\mathbb{C}}_p$ is given by

$$\hat{p}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, p(z_{-2}), p(z_{-1}), p(z_0))$$

The inverse erases the last entry

$\check{p} : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p$ is given by

$$\hat{p}(x, n) = (f(x), n) = (x, n+1)$$

The inverse is $(x, n) \mapsto (x, n-1)$, i.e., it erases the first entry.

Theorem 1 (H, Oberste-Vorth). *For all hyperbolic polynomials p with K_p connected, there exists $\varepsilon > 0$ such that if $0 < |b| < \varepsilon$, there exist homeomorphisms*

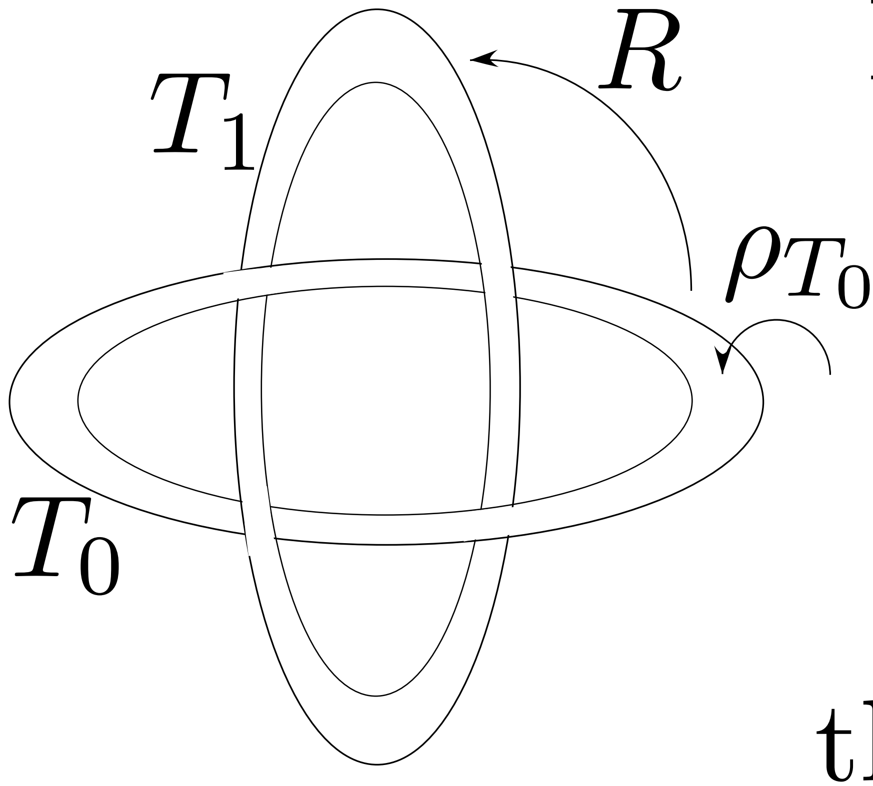
$$\Phi^- : \widehat{\mathbb{C}}_p \to J^-_{p,b}, \quad \Phi^+ : \widehat{\mathbb{C}}_p \to J^+_{p,b}$$

such that the diagrams

$$\begin{tikzcd} \widehat{\mathbb{C}}_p \arrow[r, "\Phi^-"] \arrow[d, "\hat{p}"'] & J^-_{p,b} \arrow[d, "H_{p,b}"] \\ \widehat{\mathbb{C}}_p \arrow[r, "\Phi^-"] & J^-_{p,b} \end{tikzcd} \qquad \begin{tikzcd} \check{\mathbb{C}}_p \arrow[r, "\Phi^+"] \arrow[d, "\check{p}"'] & J^+_{p,b} \arrow[d, "H_{p,b}"] \\ \check{\mathbb{C}}_p \arrow[r, "\Phi^+"] & J^+_{p,b} \end{tikzcd}$$

commute.

Until recently, there was only one case where we understood how to place these objects in $\mathbb{R}^4$.



Describing this situation requires linked solenoid mappings.

If T is an unknotted torus in S^3, there exists a homeomorphism ρ_T that maps the inside to the outside and the outside to the inside.

Take two tori T_0 and T_1 linked with linking number d in the simplest way. Let R rotate T_0 to T_1.

Then $\rho_{T_0} \circ R$ is a linked solenoid map of degree d.

Note that R maps the inside of T_0 to the inside of T_1, hence completely outside T_0.

Note that R maps the inside of T_0 to the inside of T_1, hence completely outside T_0.

Thus

Note that R maps the inside of T_0 to the inside of T_1, hence completely outside T_0.

Thus

$$\sigma(T_0) = \rho_{T_0} \circ R(T_0) = \rho_{T_0}(T_1) \subset T_0$$

Note that R maps the inside of T_0 to the inside of T_1, hence completely outside T_0.

Thus

$$\sigma(T_0) = \rho_{T_0} \circ R(T_0) = \rho_{T_0}(T_1) \subset T_0$$

So σ maps T_0 inside itself; it is easy to see that it is a "solenoidal mapping."

Note that R maps the inside of T_0 to the inside of T_1, hence completely outside T_0.

Thus

$$\sigma(T_0) = \rho_{T_0} \circ R(T_0) = \rho_{T_0}(T_1) \subset T_0$$

So σ maps T_0 inside itself; it is easy to see that it is a "solenoidal mapping."

There is an attracting solenoid Σ^+ contained in

Note that R maps the inside of T_0 to the inside of T_1, hence completely outside T_0.

Thus

$$\sigma(T_0) = \rho_{T_0} \circ R(T_0) = \rho_{T_0}(T_1) \subset T_0$$

So σ maps T_0 inside itself; it is easy to see that it is a "solenoidal mapping."

There is an attracting solenoid Σ^+ contained in T_0

Note that R maps the inside of T_0 to the inside of T_1, hence completely outside T_0.

Thus

$$\sigma(T_0) = \rho_{T_0} \circ R(T_0) = \rho_{T_0}(T_1) \subset T_0$$

So σ maps T_0 inside itself; it is easy to see that it is a "solenoidal mapping."

There is an attracting solenoid Σ^+ contained in T_0

and a repelling solenoid Σ^- contained in T_1

Describe $\mathbb{R}^4$ by polar coordinates $r \geq 0, \theta \in S^3$.

Describe $\mathbb{R}^4$ by polar coordinates $r \geq 0, \theta \in S^3$.

Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by

Describe $\mathbb{R}^4$ by polar coordinates $r \geq 0, \theta \in S^3$.

Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$F(r,\theta) = \big(r^2, \sigma(\theta)\big)$$

Describe $\mathbb{R}^4$ by polar coordinates $r \geq 0, \theta \in S^3$.

Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$F(r, \theta) = \big(r^2, \sigma(\theta)\big)$$

Let

Describe $\mathbb{R}^4$ by polar coordinates $r \geq 0, \theta \in S^3$.

Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$F(r, \theta) = \big(r^2, \sigma(\theta)\big)$$

Let

$$X = \mathbb{R}^4 - \{(r, \theta) \mid r \geq 1, \theta \in \Sigma^-\}$$

Describe $\mathbb{R}^4$ by polar coordinates $r \geq 0, \theta \in S^3$.

Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$F(r, \theta) = \big(r^2, \sigma(\theta)\big)$$

Let

$$X = \mathbb{R}^4 - \{(r, \theta) \mid r \geq 1, \theta \in \Sigma^-\}$$

The space X is homeomorphic to $\mathbb{R}^4$, and F-invariant.

Describe $\mathbb{R}^4$ by polar coordinates $r \geq 0, \theta \in S^3$.

Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$F(r, \theta) = \big(r^2, \sigma(\theta)\big)$$

Let

$$X = \mathbb{R}^4 - \{(r, \theta) \mid r \geq 1, \theta \in \Sigma^-\}$$

The space X is homeomorphic to $\mathbb{R}^4$, and F-invariant.

The mapping $F : X \to X$ is our model.

Theorem 2 (Sylvain Bonnot). *For all polynomials p having an attracting fixed point with all the critical points in its immediate domain, there exists $\varepsilon > 0$ such that if $0 < |b| < \varepsilon$, then there exists a homeomorphism $\Phi : X \to \mathbb{C}^2$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & \mathbb{C}^2 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle H_{p,b}} \\ X & \xrightarrow{\Phi} & \mathbb{C}^2 \end{array}$$

commutes.

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

Recall that if p is monic and K_p is connected, there is a unique Böttcher coordinate

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

Recall that if p is monic and K_p is connected, there is a unique Böttcher coordinate

$$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p$$

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

Recall that if p is monic and K_p is connected, there is a unique Böttcher coordinate

$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p$ such that

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

Recall that if p is monic and K_p is connected, there is a unique Böttcher coordinate

$$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p \quad \text{such that} \quad \psi_p(z^d) = p(\psi_p(z)).$$

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

Recall that if p is monic and K_p is connected, there is a unique Böttcher coordinate

$$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p \quad \text{such that} \quad \psi_p(z^d) = p(\psi_p(z)).$$

If K_p is in addition locally connected, ψ extends to the unit circle, and we can define

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

Recall that if p is monic and K_p is connected, there is a unique Böttcher coordinate

$$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p \quad \text{such that} \quad \psi_p(z^d) = p(\psi_p(z)).$$

If K_p is in addition locally connected, ψ extends to the unit circle, and we can define the Caratheodory loop γ_p by the formula

This model is already rather non-trivial, but the Hénon maps conjugate to the Bonnot model are the simplest Hénon mappings, the ones to which all others are compared.

They correspond to the polynomials $z \mapsto z^d$.

Recall that if p is monic and K_p is connected, there is a unique Böttcher coordinate

$$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p \quad \text{such that} \quad \psi_p(z^d) = p(\psi_p(z)).$$

If K_p is in addition locally connected, ψ extends to the unit circle, and we can define the Caratheodory loop γ_p by the formula

$$\gamma_p(t) = \psi_p(e^{2\pi i t}) = \lim_{r \searrow 1} \psi_p(r e^{2\pi i t}).$$

This map $\gamma_p : \mathbb{R}/\mathbb{Z} \to K_p$ induces an equivalence relation on S^1 by

This map $\gamma_p : \mathbb{R}/\mathbb{Z} \to K_p$ induces an equivalence relation on S^1 by

$$t_1 \sim_p t_2 \quad \Longleftrightarrow \quad \gamma_p(t_1) = \gamma_p(t_2).$$

This map $\gamma_p : \mathbb{R}/\mathbb{Z} \to K_p$ induces an equivalence relation on S^1 by

$$t_1 \sim_p t_2 \qquad \Longleftrightarrow \qquad \gamma_p(t_1) = \gamma_p(t_2).$$

The convex hulls of the equivalence classes form the invariant lamination of p.

This map $\gamma_p : \mathbb{R}/\mathbb{Z} \to K_p$ induces an equivalence relation on S^1 by

$$t_1 \sim_p t_2 \qquad \Longleftrightarrow \qquad \gamma_p(t_1) = \gamma_p(t_2).$$

The convex hulls of the equivalence classes form the invariant lamination of p.

The quotient of $\mathbb{C}$ by $\sim_p$ is a topological model for $(\mathbb{C}, K_p)$, called the pinched disk model.

This map $\gamma_p : \mathbb{R}/\mathbb{Z} \to K_p$ induces an equivalence relation on S^1 by

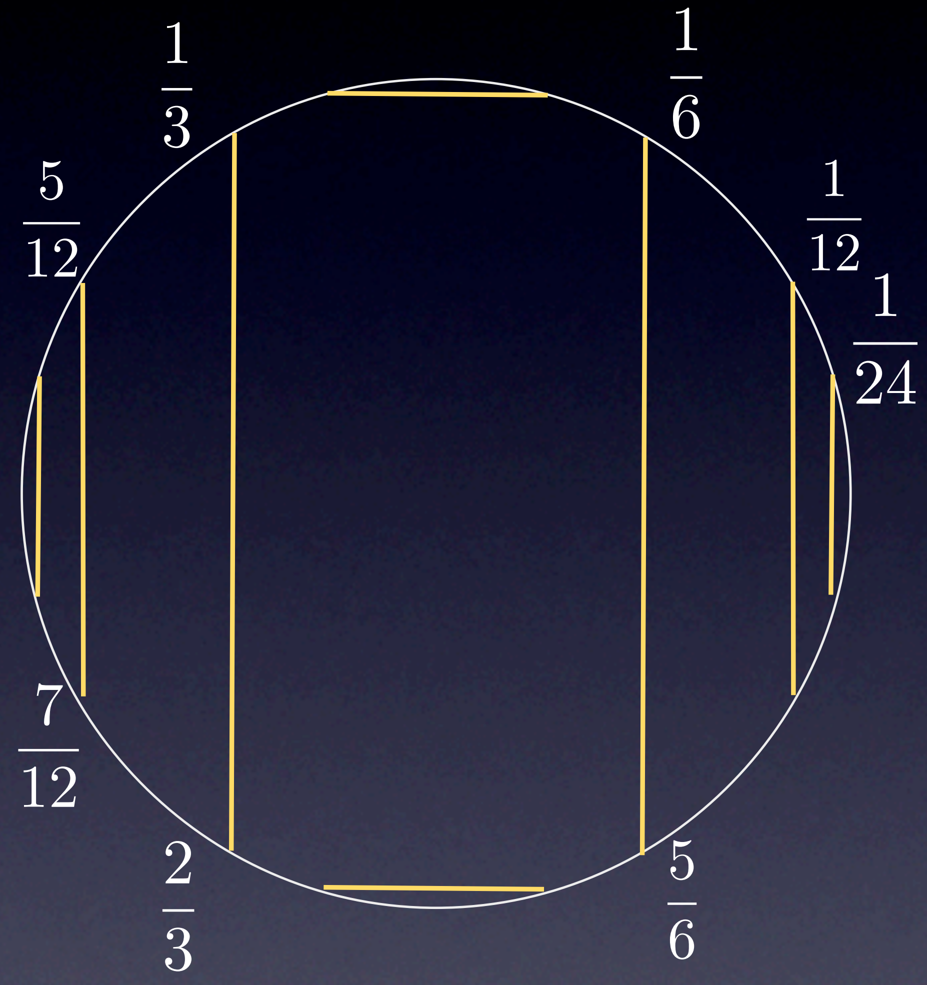
$$t_1 \sim_p t_2 \quad \iff \quad \gamma_p(t_1) = \gamma_p(t_2).$$

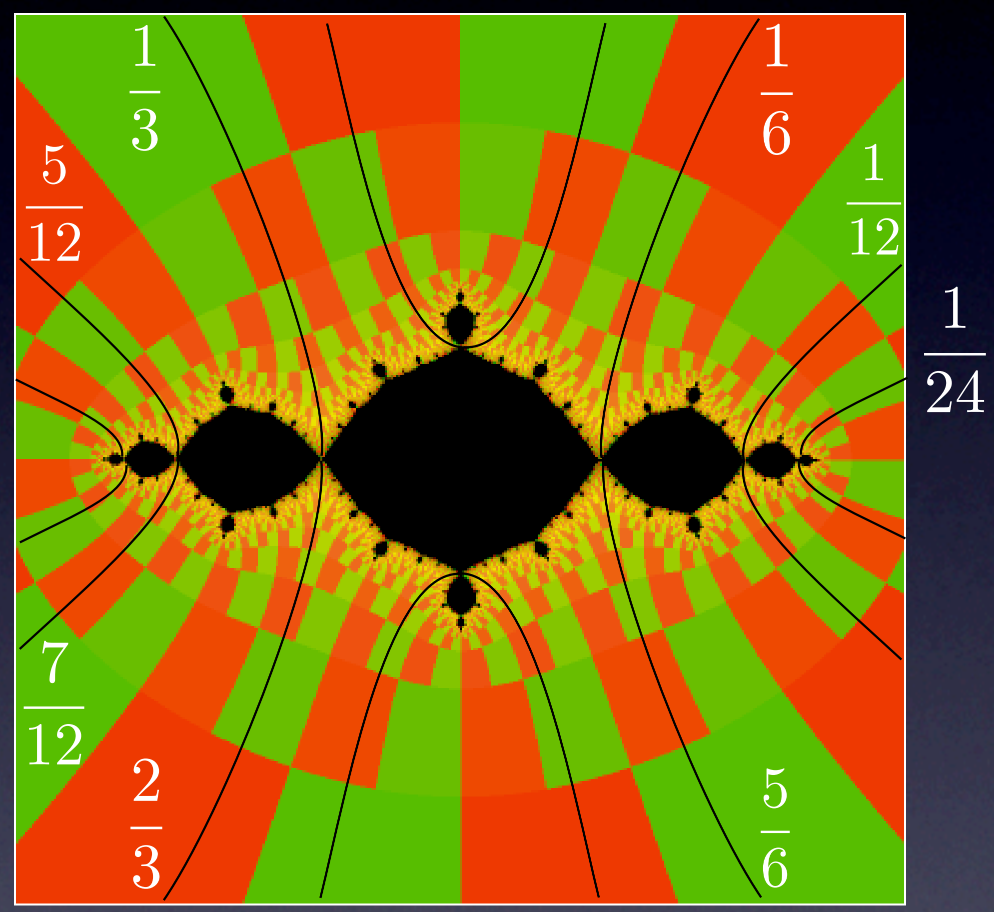
The convex hulls of the equivalence classes form the invariant lamination of p.

The quotient of $\mathbb{C}$ by $\sim_p$ is a topological model for $(\mathbb{C}, K_p)$, called the pinched disk model.

Our aim is to provide a model for (some) Hénon mappings, by pinching the ball of the Bonnot model.

Rather than treat a general case, we will try to understand the equivalence relation for the small perturbations of $p : z \mapsto z^2 - 1$.





In this case the invariant lamination of p is especially simple.

The corresponding equivalence relation in the ball model is an injective map

$$\Psi : \mathbb{R}^2 \times [0,1] \to B$$

The corresponding equivalence relation in the ball model is an injective map

$$\Psi : \mathbb{R}^2 \times [0,1] \to B$$

such that $\Psi(\mathbb{R}^2 \times \{0\})$ is the stable manifold of

$$\ldots \overline{10}.\overline{10} \cdots \in \Sigma^+$$

and $\Psi(\mathbb{R}^2 \times \{1\})$ is the stable manifold of

$$\ldots \overline{01}.\overline{01} \cdots \in \Sigma^+$$

The corresponding equivalence relation in the ball model is an injective map

$$\Psi : \mathbb{R}^2 \times [0,1] \to B$$

such that $\Psi(\mathbb{R}^2 \times \{0\})$ is the stable manifold of

$$\ldots \overline{10}.\overline{10} \cdots \in \Sigma^+$$

and $\Psi(\mathbb{R}^2 \times \{1\})$ is the stable manifold of

$$\ldots \overline{01}.\overline{01} \cdots \in \Sigma^+$$

The equivalence classes are all $\{x\} \times [0,1]$ for $x \in \mathbb{R}^2$.

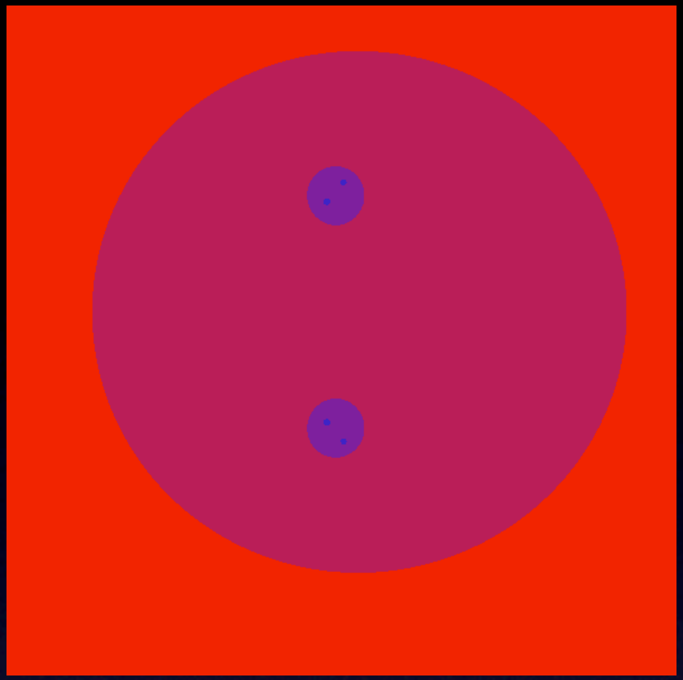
Let $\sim_\Psi$ be the equivalence relation.

Not quite a theorem [Radu and Tanase]

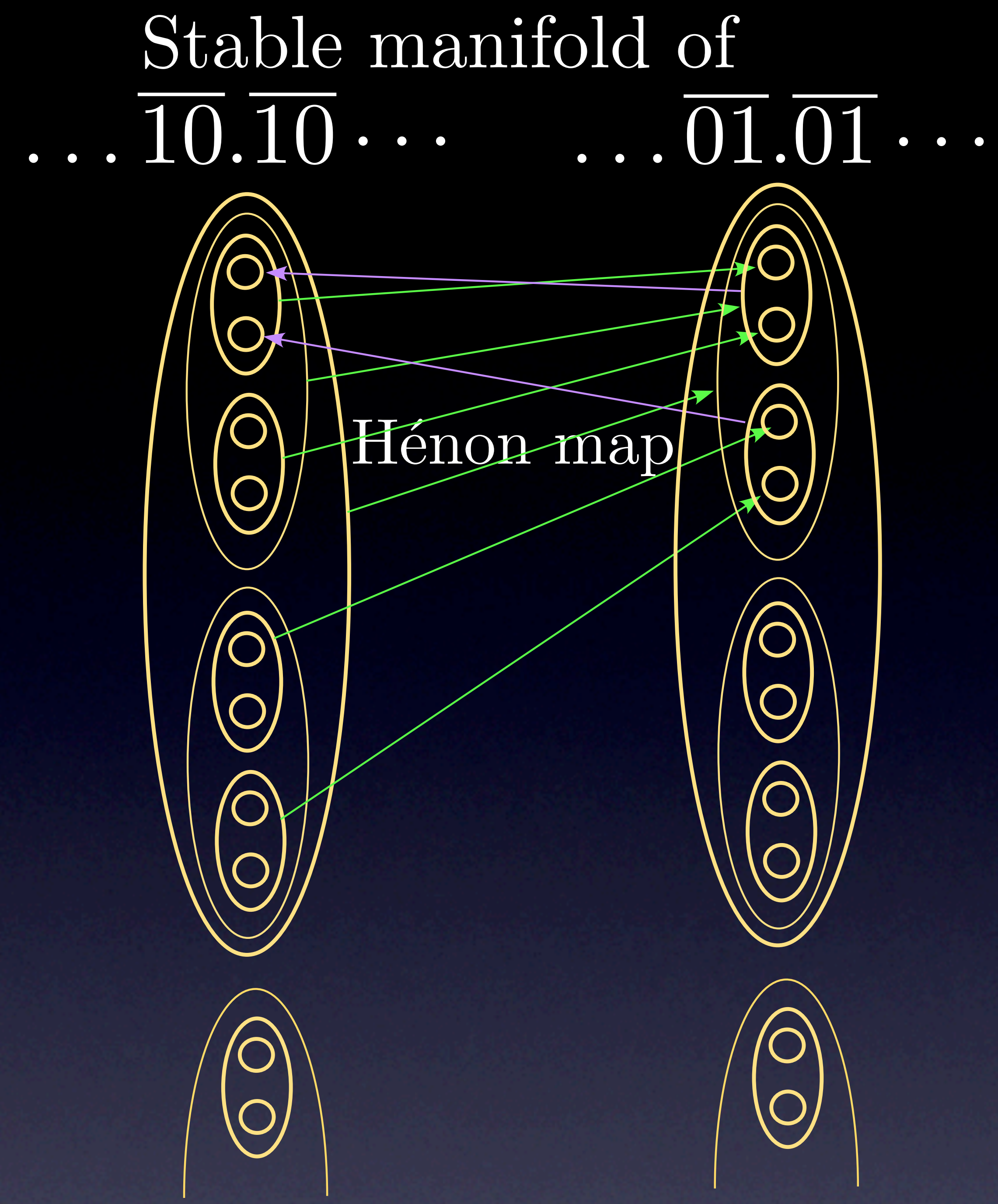
There exists such a mapping Ψ *so that for all* c *satisfying* $|c-1|<1/4$, *there exists* $\varepsilon>0$ *such that when* $|b|<\varepsilon$, *there exists a homeomorphism*

$$\Phi : X/\sim_\Psi \to \mathbb{C}^2$$

conjugating the dynamics on the model to the $H_{z^2+c,b}$.



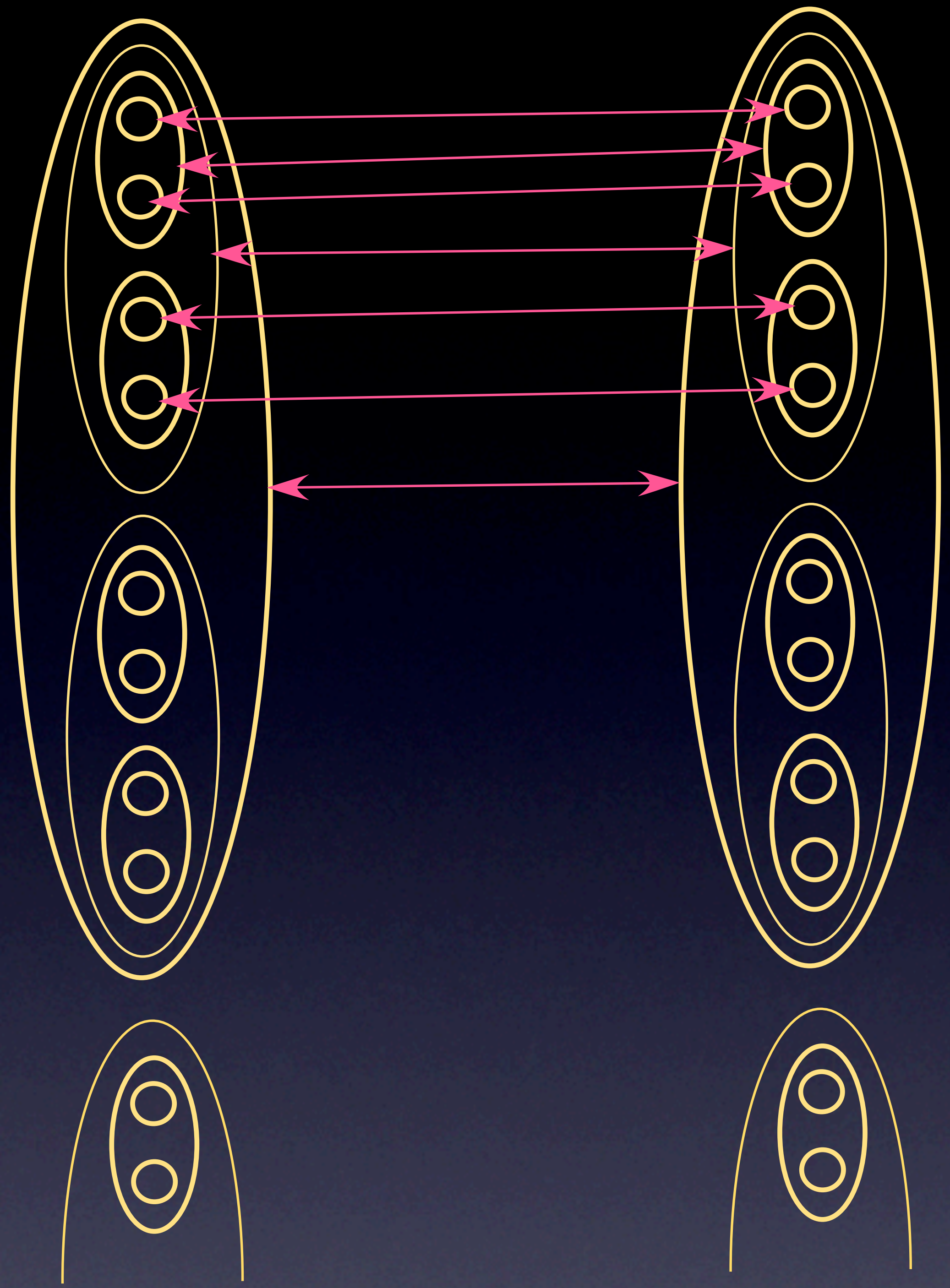
The stable manifolds really look like this.



But structurally they look like this.

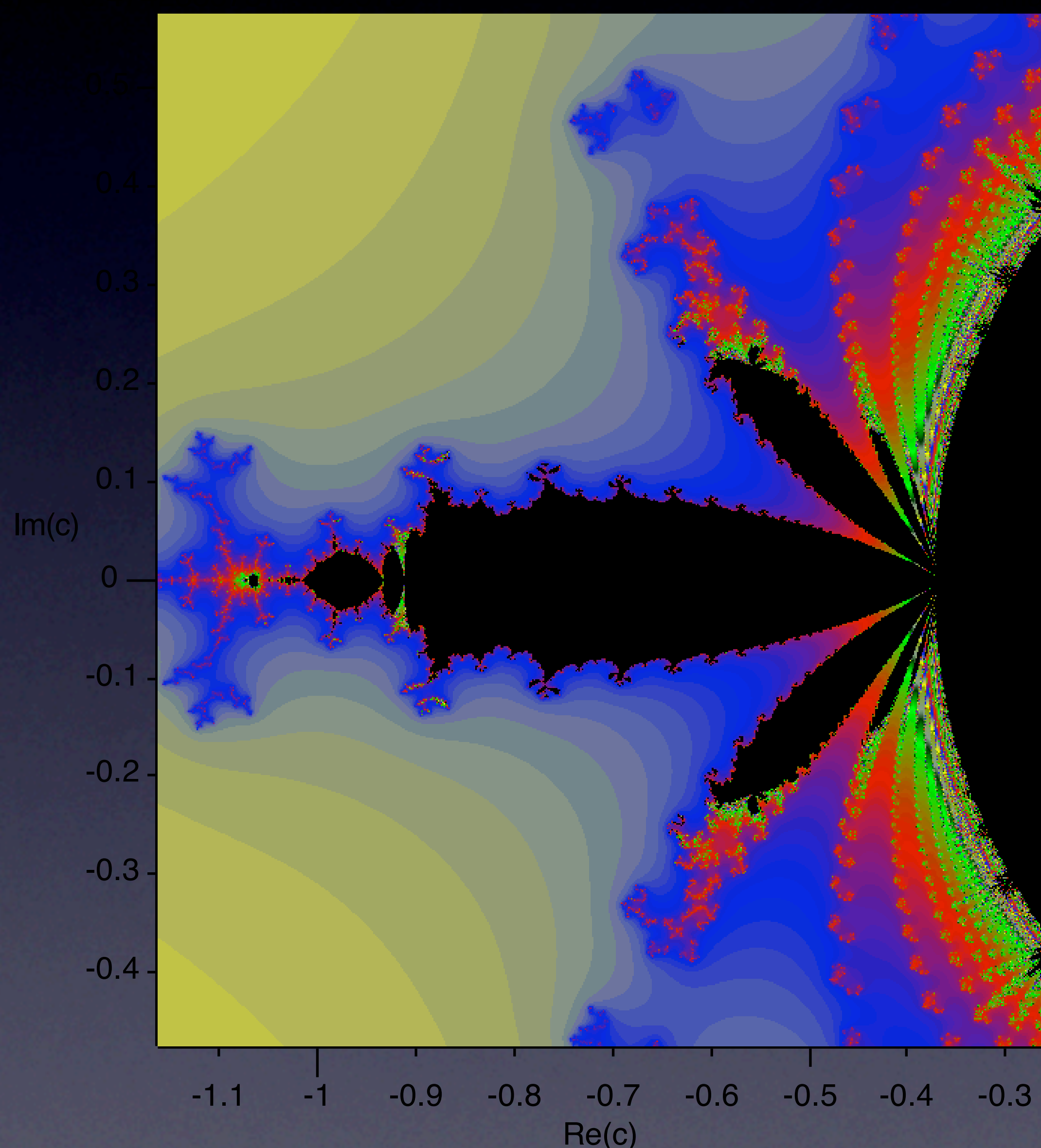
The curves are the intersections with the tori $\sigma^{\circ n}(T_0)$.

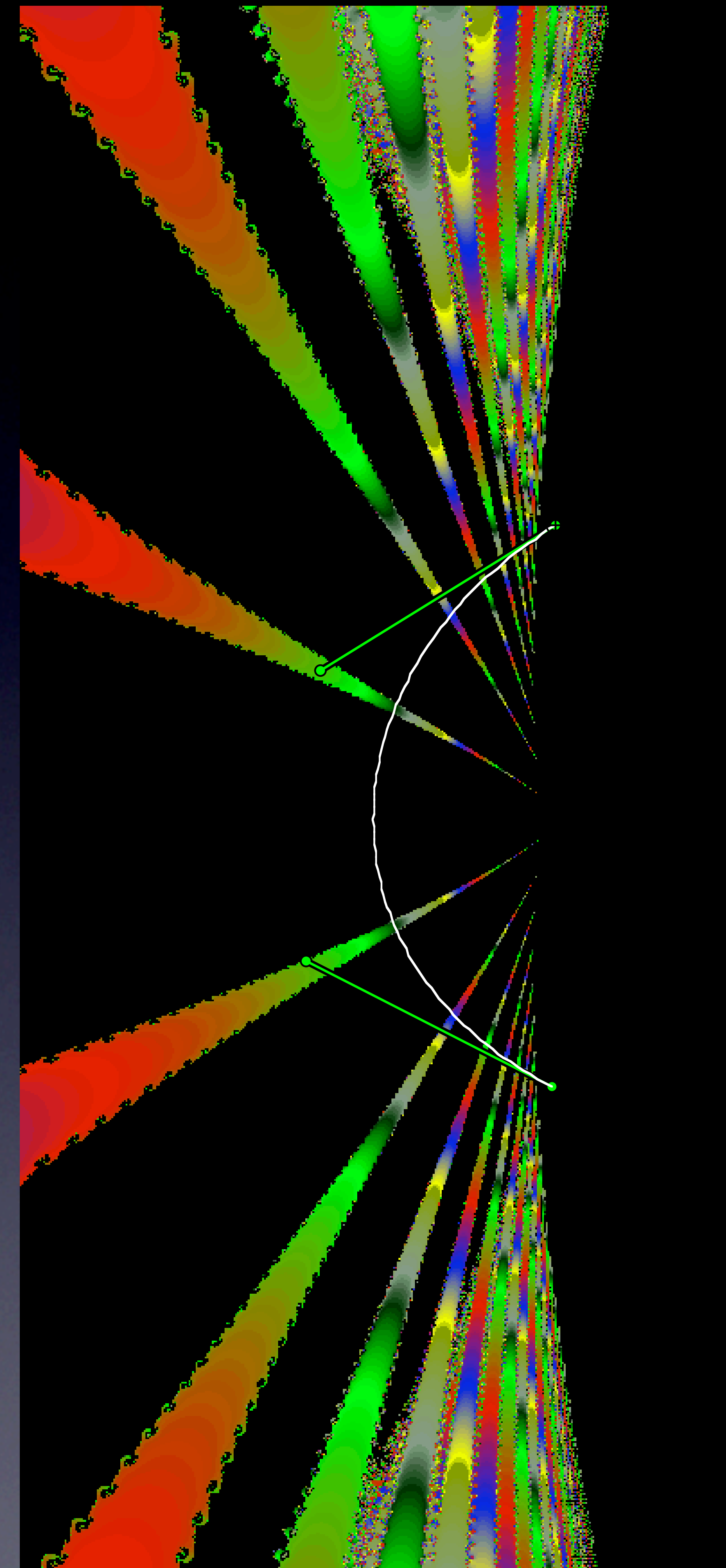
The identifications identify points at the same level, and must be compatible with the dynamics.

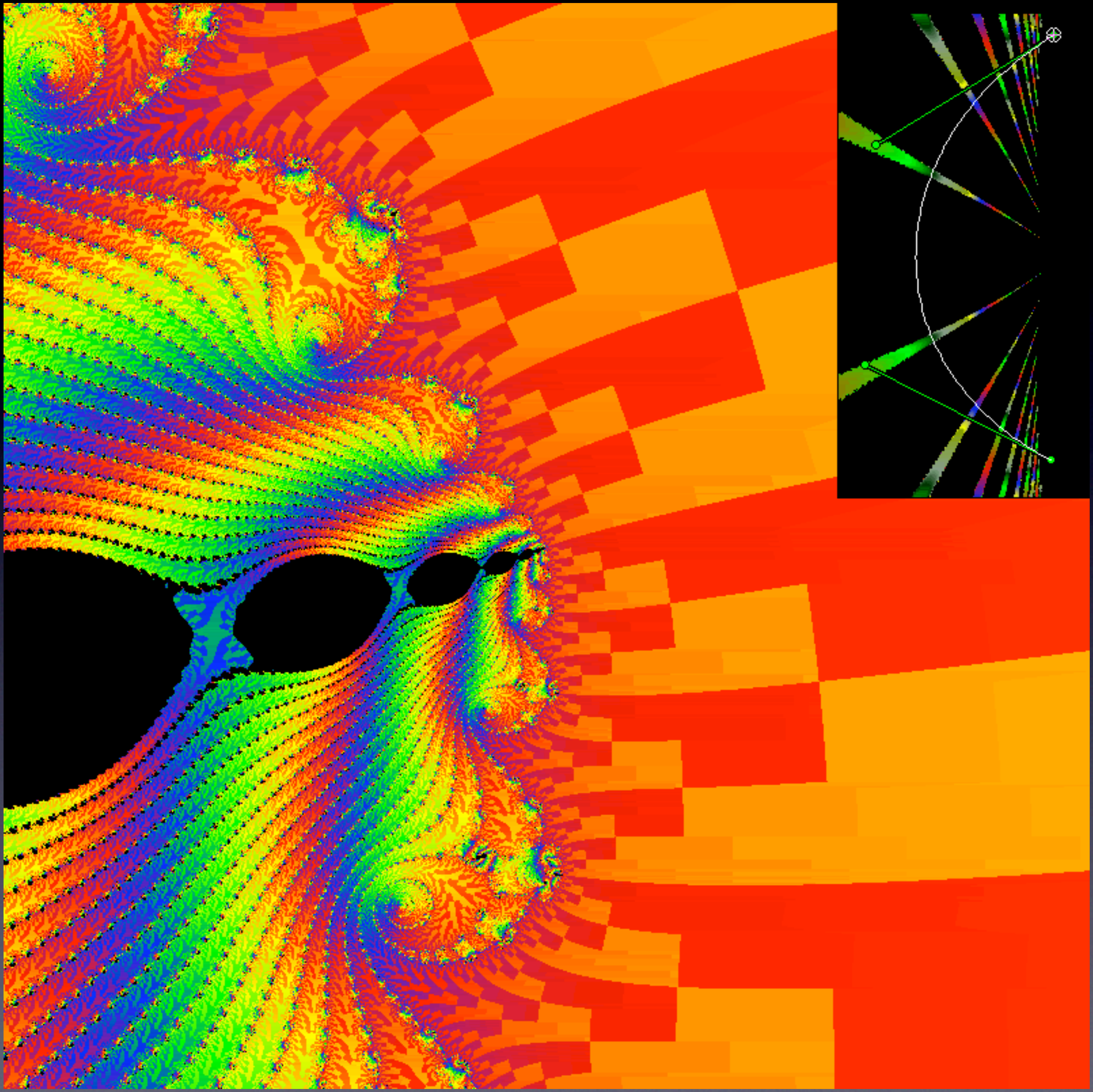


But there are other identifications possible.
Look at the following picture of parameter space,
and the corresponding movie.

Parameter space for Jacobian = -.3







We think we know how to construct a pinched ball model for all the “fingers”

We think we know how to construct a pinched ball model for all the "fingers" without modifying the image of Ψ, but by changing what is connected to what in $\mathbb{R}^2 \times [0,1]$.

We think we know how to construct a pinched ball model for all the "fingers" without modifying the image of Ψ, but by changing what is connected to what in $\mathbb{R}^2 \times [0,1]$.

To prove that this is correct, we will need to perturb not from the polynomials, but from the

We think we know how to construct a pinched ball model for all the "fingers" without modifying the image of Ψ, but by changing what is connected to what in $\mathbb{R}^2 \times [0, 1]$.

To prove that this is correct, we will need to perturb not from the polynomials, but from the curve in parameter space where the Hénon map has a fixed point with one eigenvalue $= -1$.

There are a great many open questions:

Are there infinitely many fingers?

Do they all have pinched ball models?

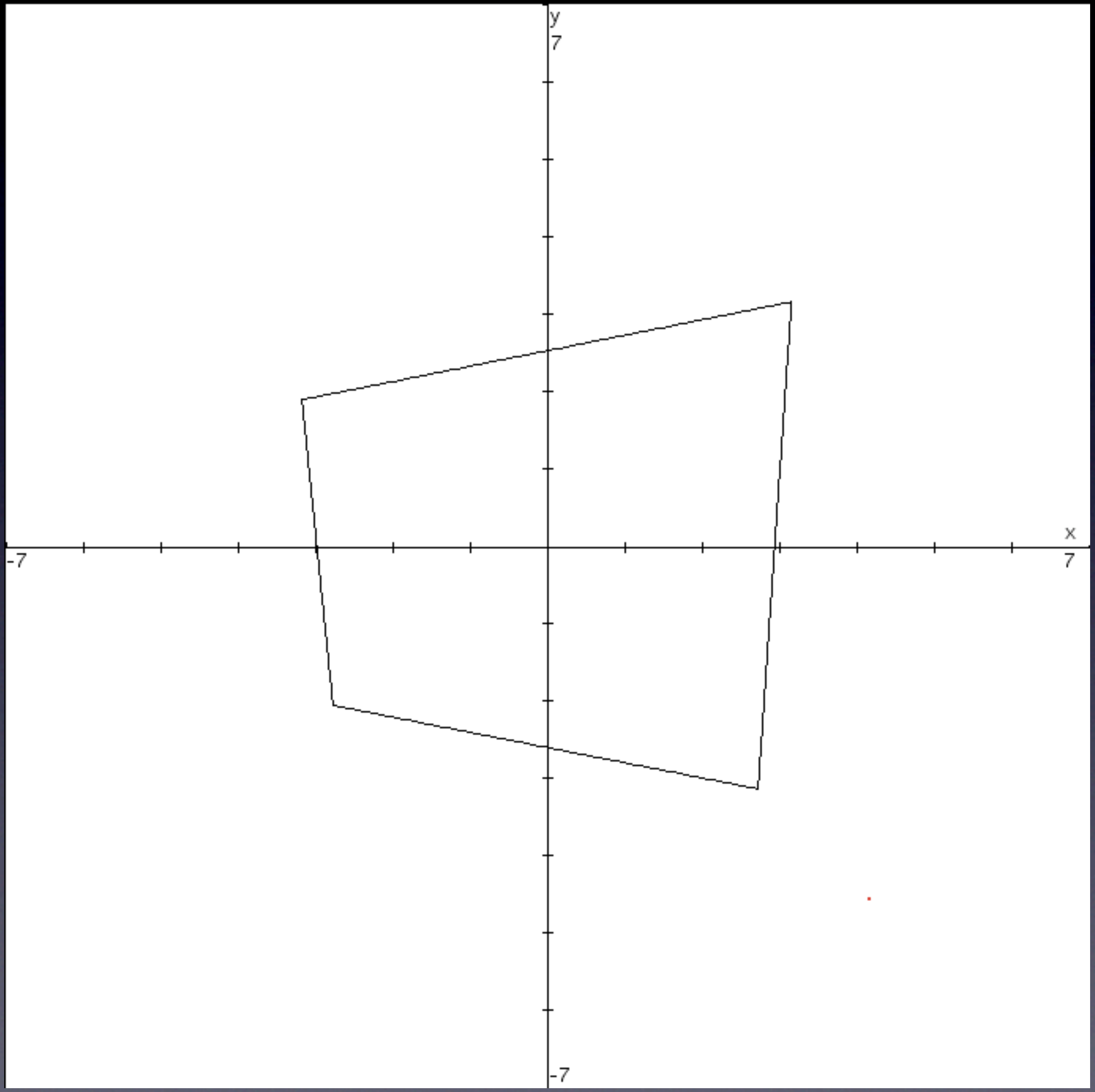
Do all bifurcations in the quadratic family exhibit the same sort of behavior?

Monodromy of horseshoes

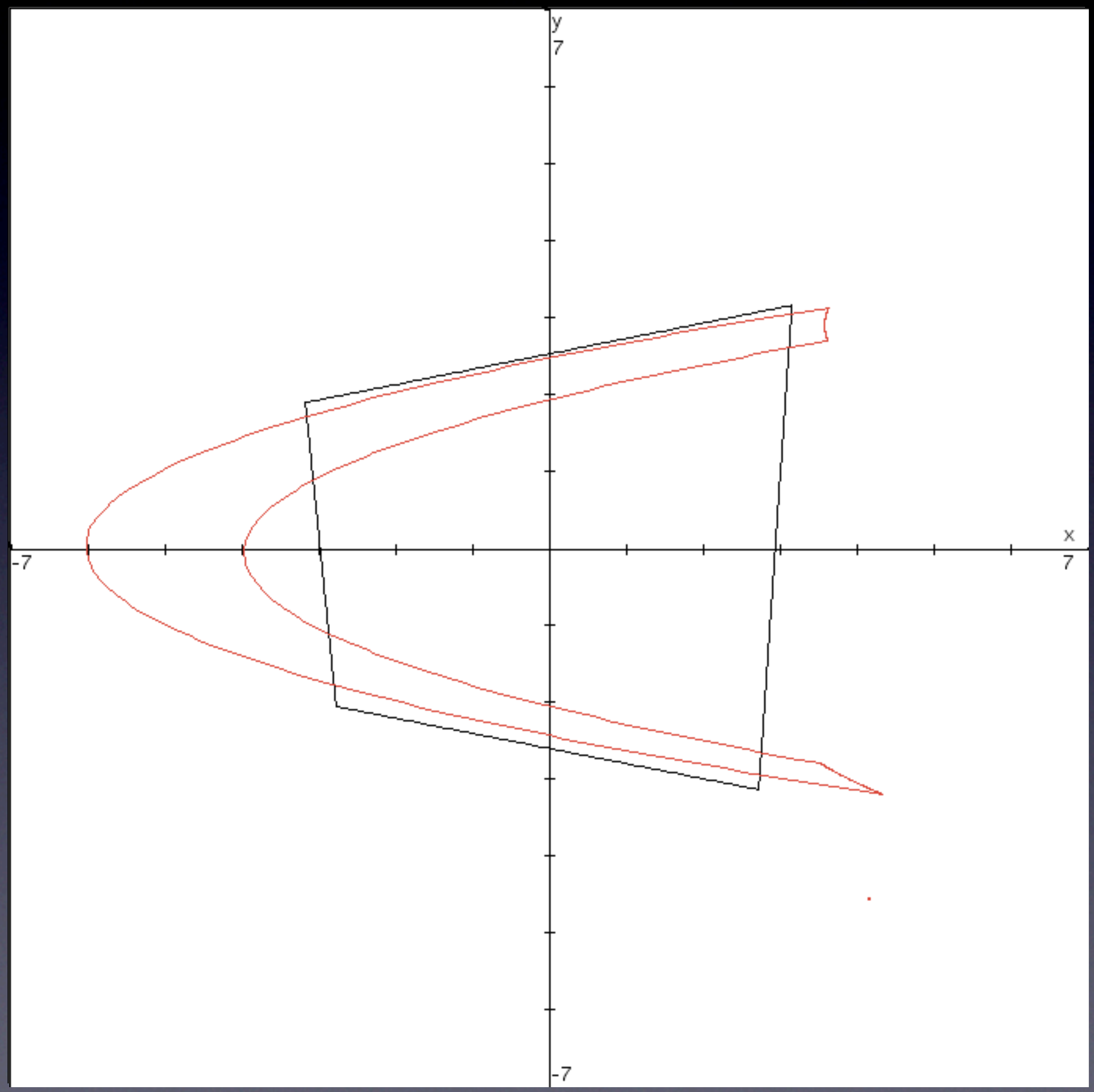
The pinched ball approach to Hénon mappings
is an "inside-out" approach.
There is also an outside-in approach,
corresponding to exploring what corresponds to the
outside of the Mandelbrot set:

The horseshoe locus

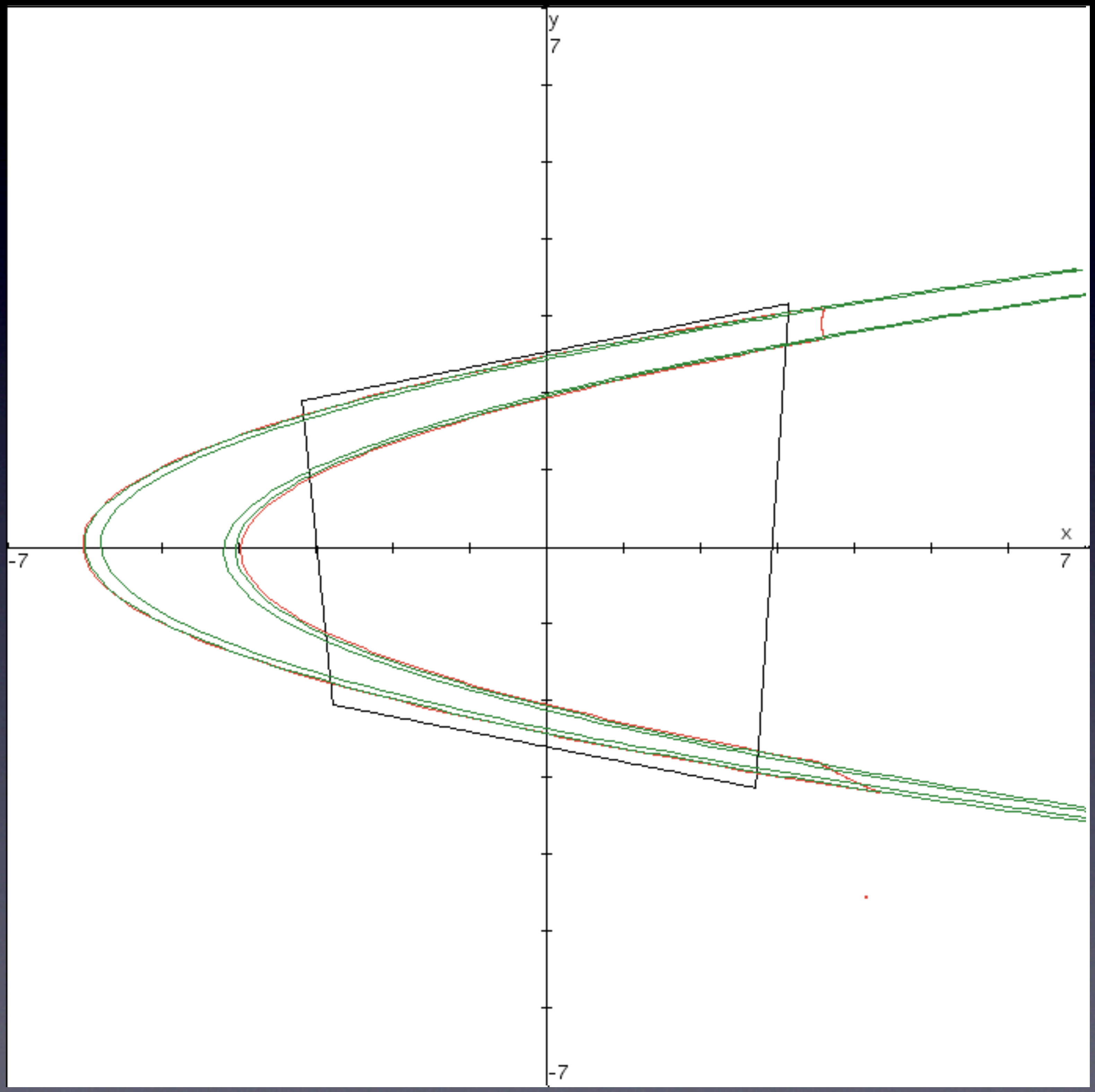
For b real and $c < -R$ with R large, the Hénon map is a Smale horseshoe.



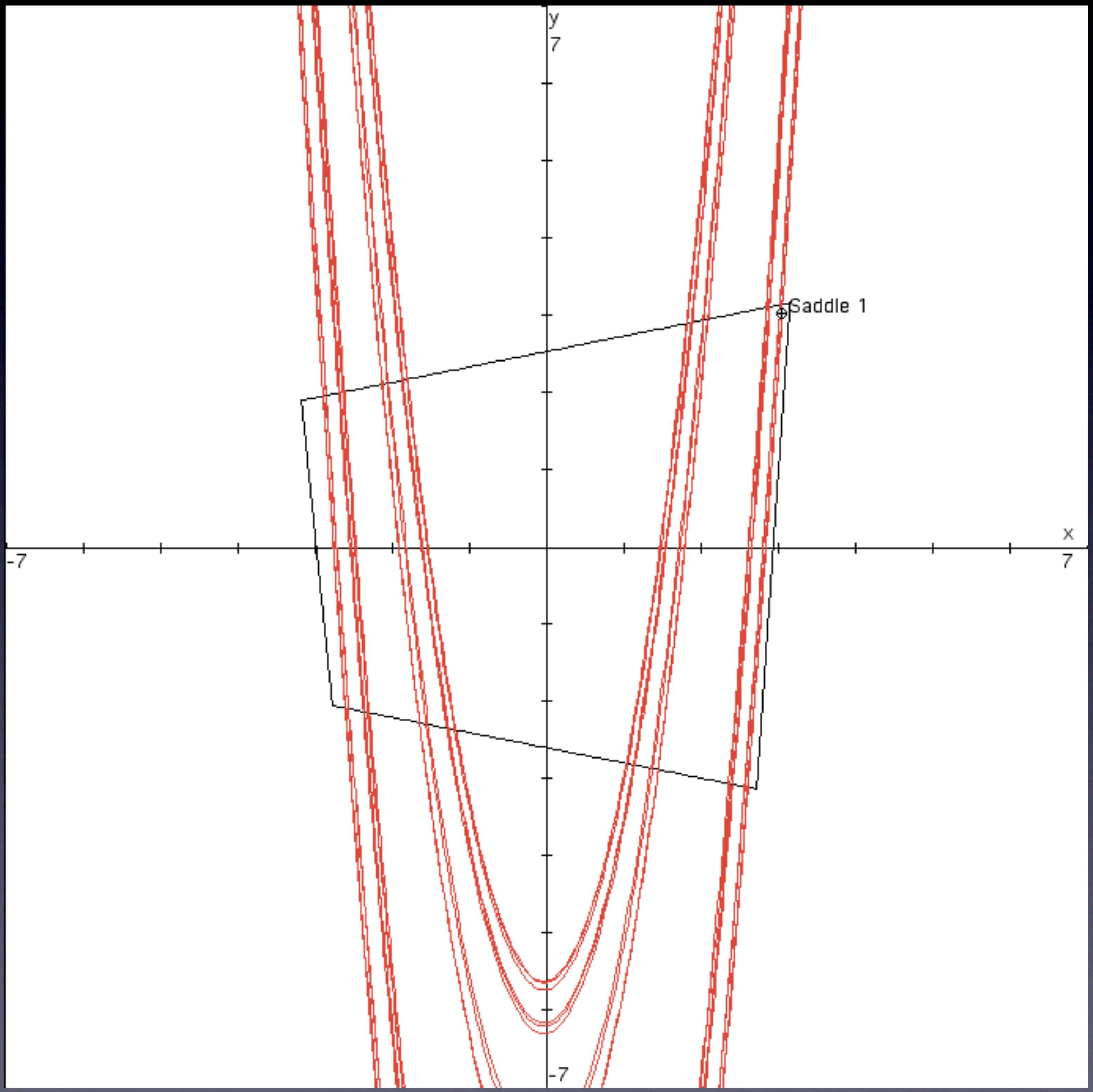
For b real and $c < -R$ with R large, the Hénon map is a Smale horseshoe.



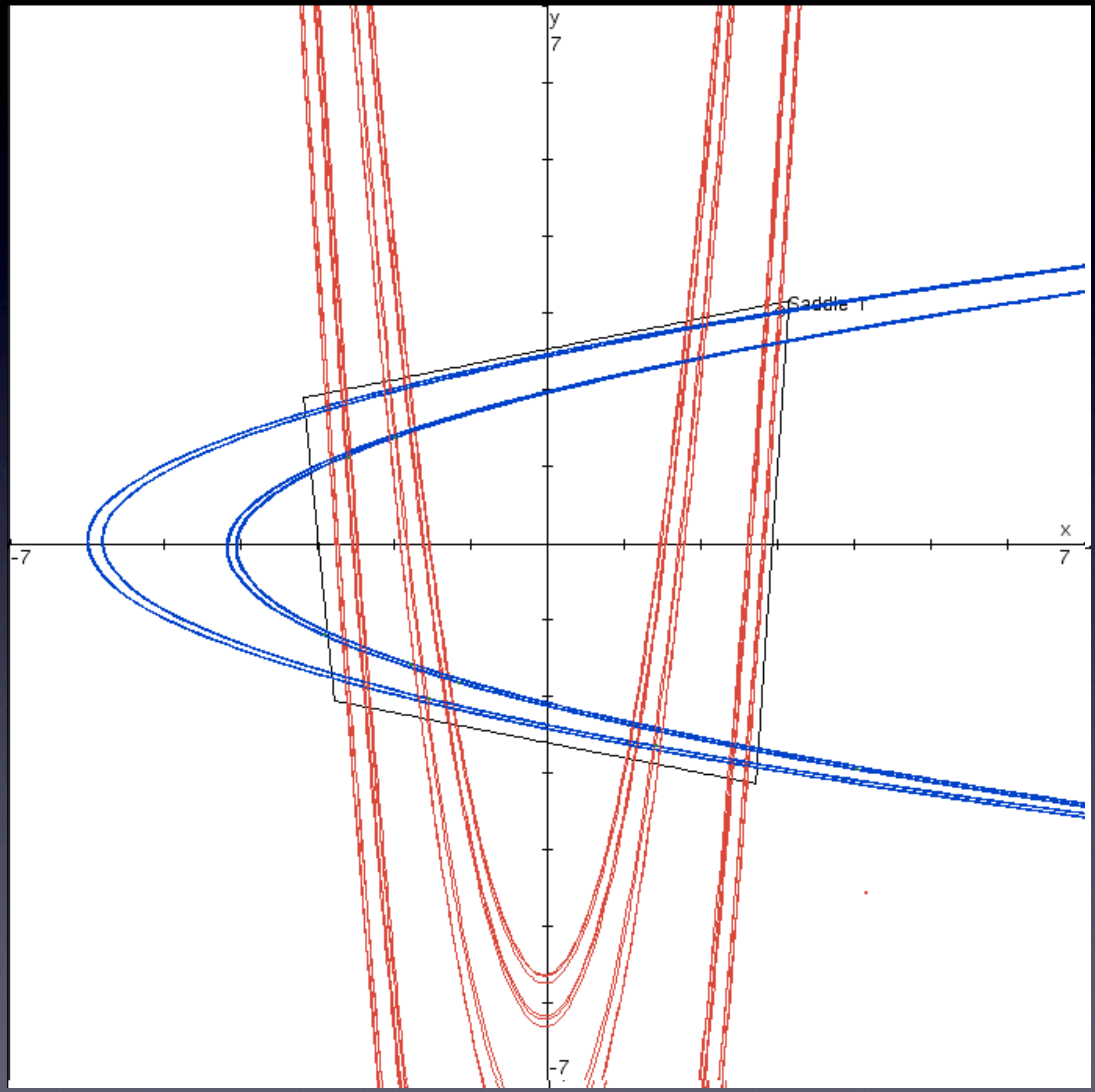
For b real and $c < -R$ with R large, the Hénon map is a Smale horseshoe.



The stable and unstable manifolds of a fixed point also describe this horseshoe.



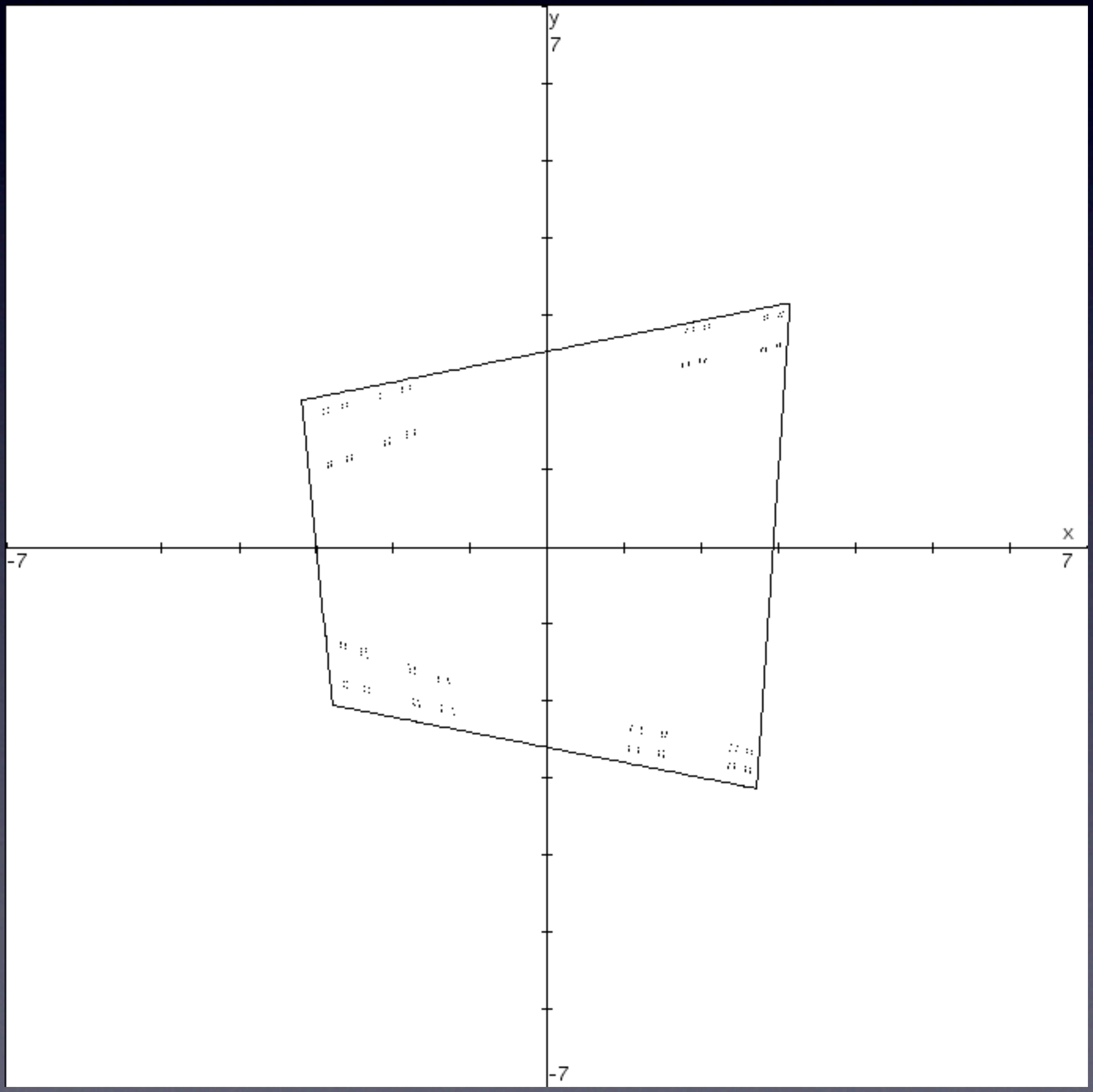
The stable and unstable manifolds of a fixed point also describe this horseshoe.



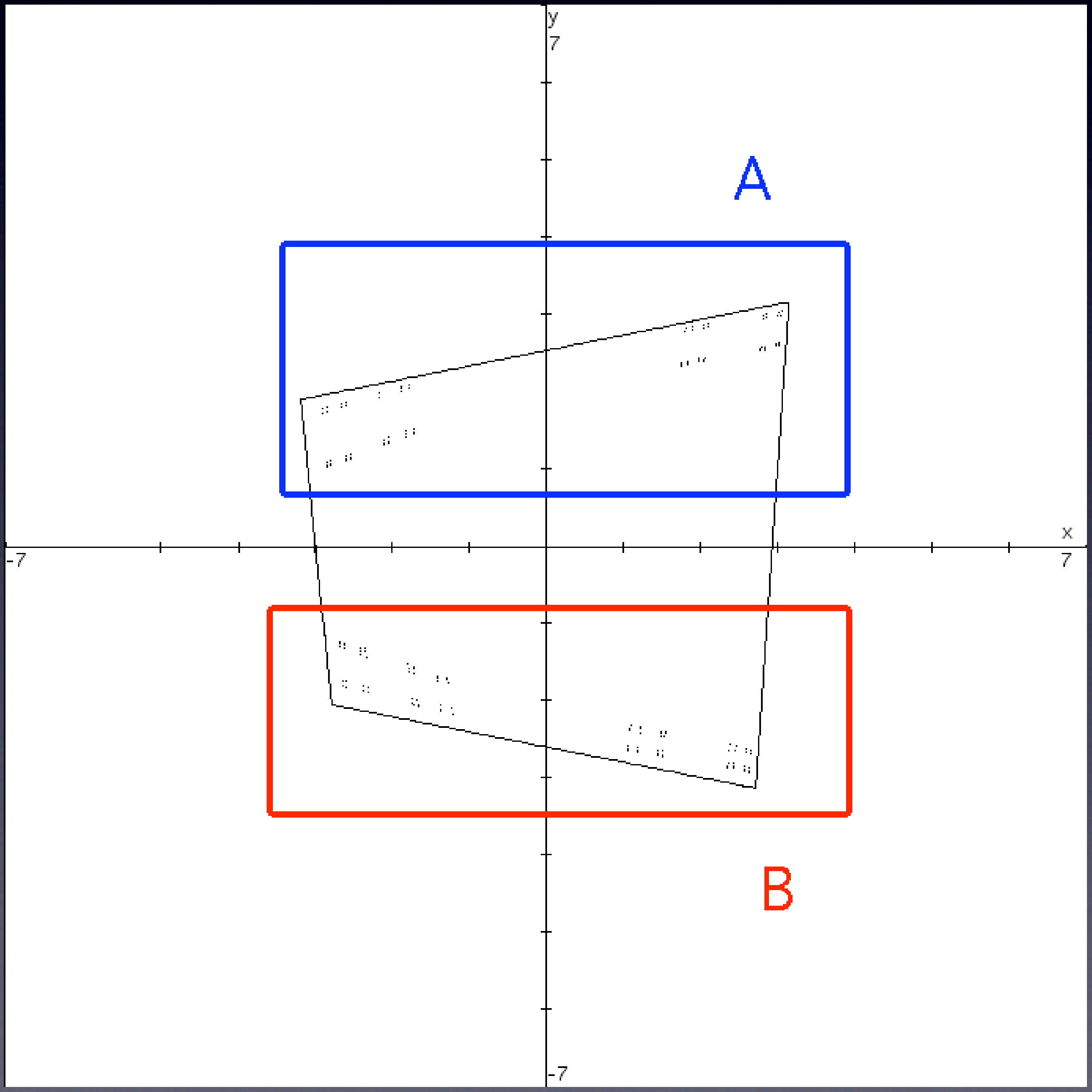
In this case, the red locus is a good approximation to $K^+ \cap \mathbb{R}^2$,

the blue locus is a good approximation to $K^- \cap \mathbb{R}^2$,

and their intersection $K = K^+ \cap K^-$ is entirely real.



This set K is homeomorphic to the full shift on 2 symbols A and B, by specifying how its orbit, backwards and forward, visits the regions A and B.



Horseshoe Locus

- The complex horseshoe locus L is the open region of parameter space where the action of the Hénon map on K set is conjugate to the horseshoe map.

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} c \\ b \end{pmatrix} \mid \begin{pmatrix} x \\ y \end{pmatrix} \in K_{c,b} \right\}$$

Horseshoe Locus

- The complex horseshoe locus L is the open region of parameter space where the action of the Hénon map on K set is conjugate to the horseshoe map.
- The set

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} c \\ b \end{pmatrix} \mid \begin{pmatrix} x \\ y \end{pmatrix} \in K_{c,b} \right\}$$

Horseshoe Locus

- The complex horseshoe locus L is the open region of parameter space where the action of the Hénon map on K set is conjugate to the horseshoe map.
- The set

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} c \\ b \end{pmatrix} \mid \begin{pmatrix} x \\ y \end{pmatrix} \in K_{c,b} \right\}$$

is a locally trivial bundle of Cantor sets over L.

Choose a base point $P_0 = \begin{pmatrix} c_0 \\ b_0 \end{pmatrix}$

in the "real horseshoe region", for instance $c_0 = -5,\ b_0 = .3$, and identify the fiber above P_0 with the full 2-shift S_2.

It follows that there is a monodromy homomorphism

$$M : \pi_1(L, P_0) \longrightarrow \mathrm{Aut}(S_2)$$

For all Jacobians b, there exists R such that if $c > R$ then $\begin{pmatrix} c \\ b \end{pmatrix} \in L$.

Hence there is at least one non-trivial element in $\pi_1(L, P_0)$, namely

$$t \mapsto \begin{pmatrix} Pe^{2\pi i t} \\ b_0 \end{pmatrix}$$

One might wonder whether the analog of

M connected

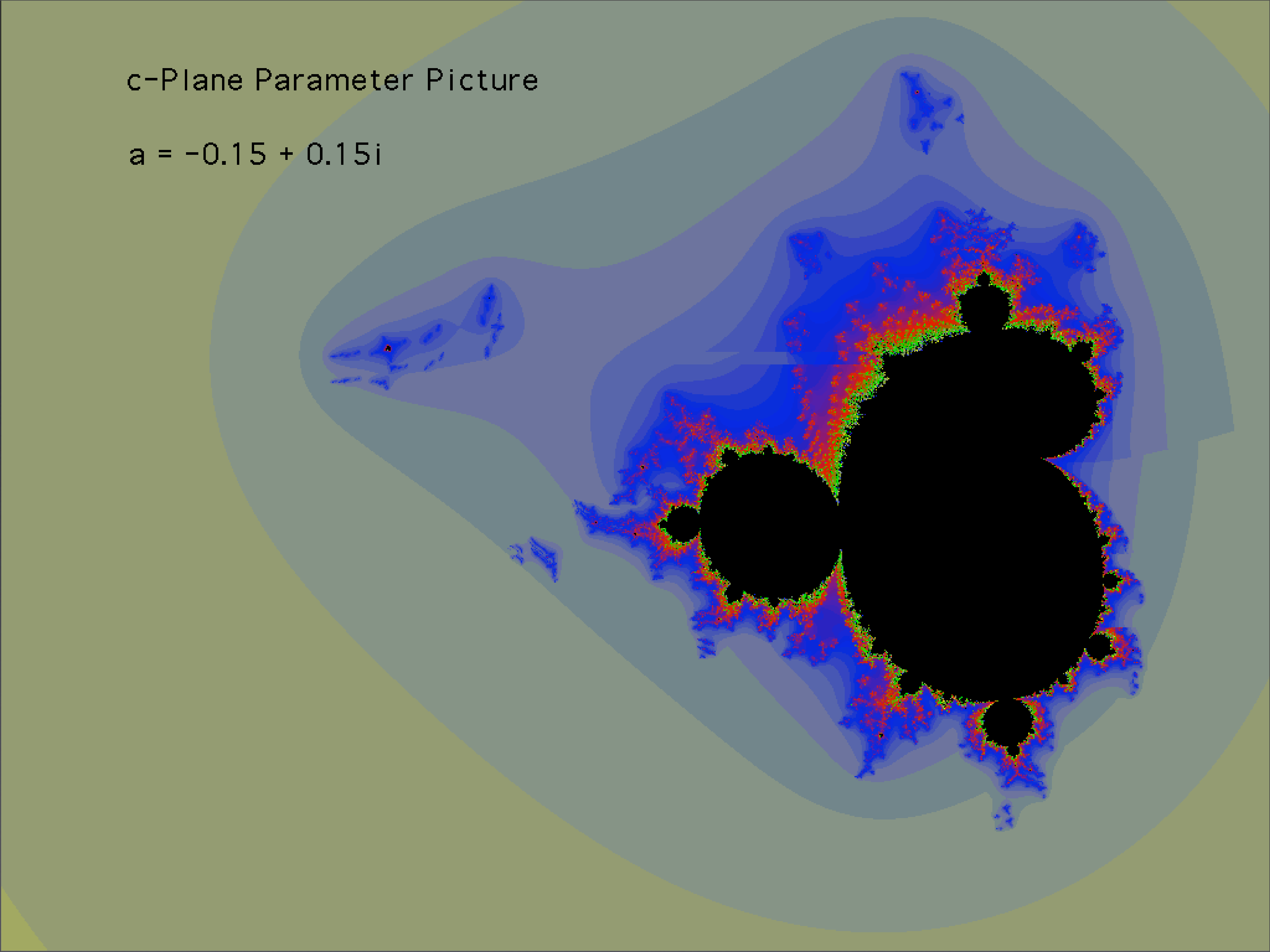
is true: is this loop the only one?

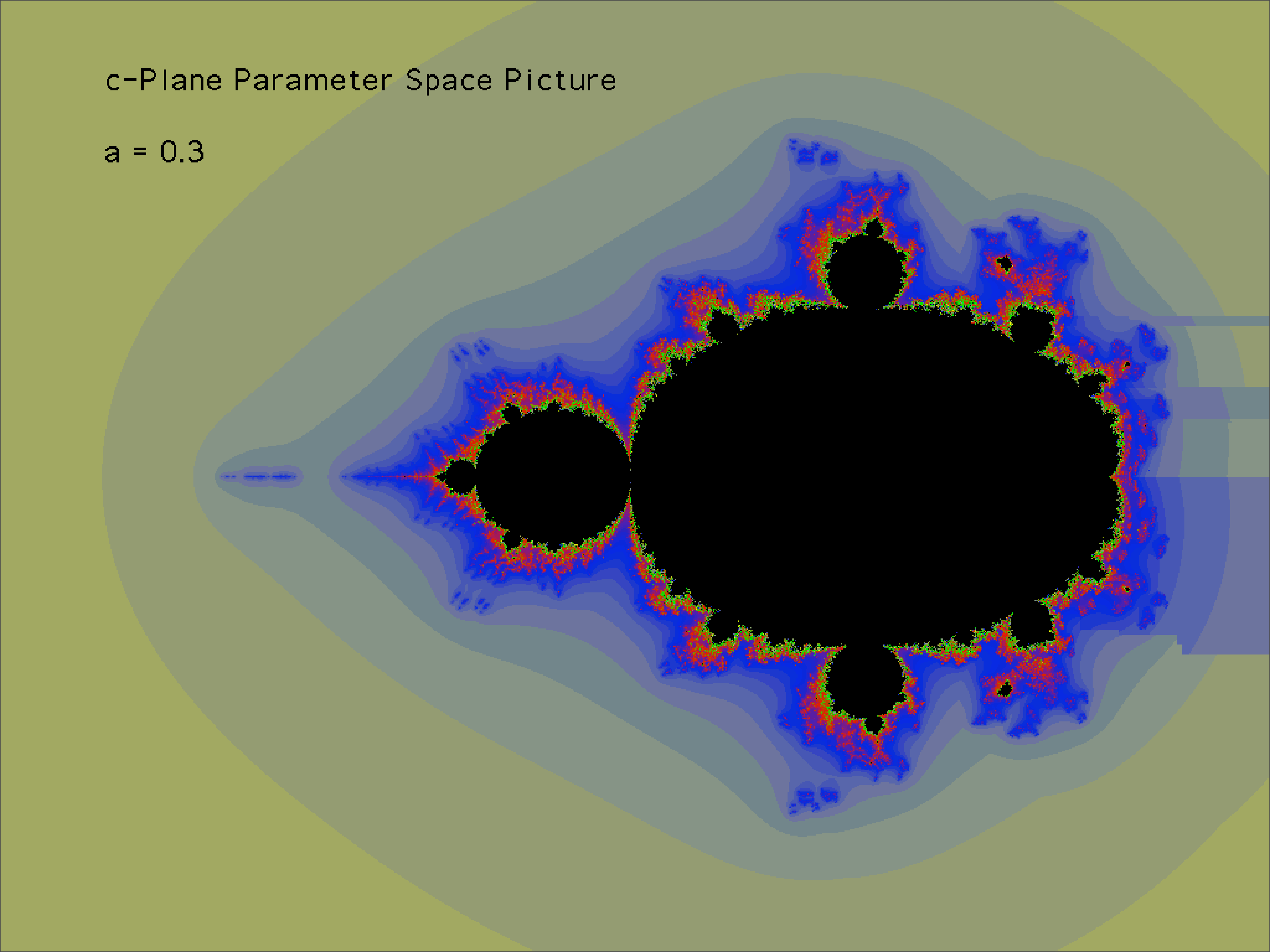
Zin Arai proved this false by exhibiting several other non-trivial loops and calculating their monodromies.

Here are some slices of parameter space
for Hénon mappings

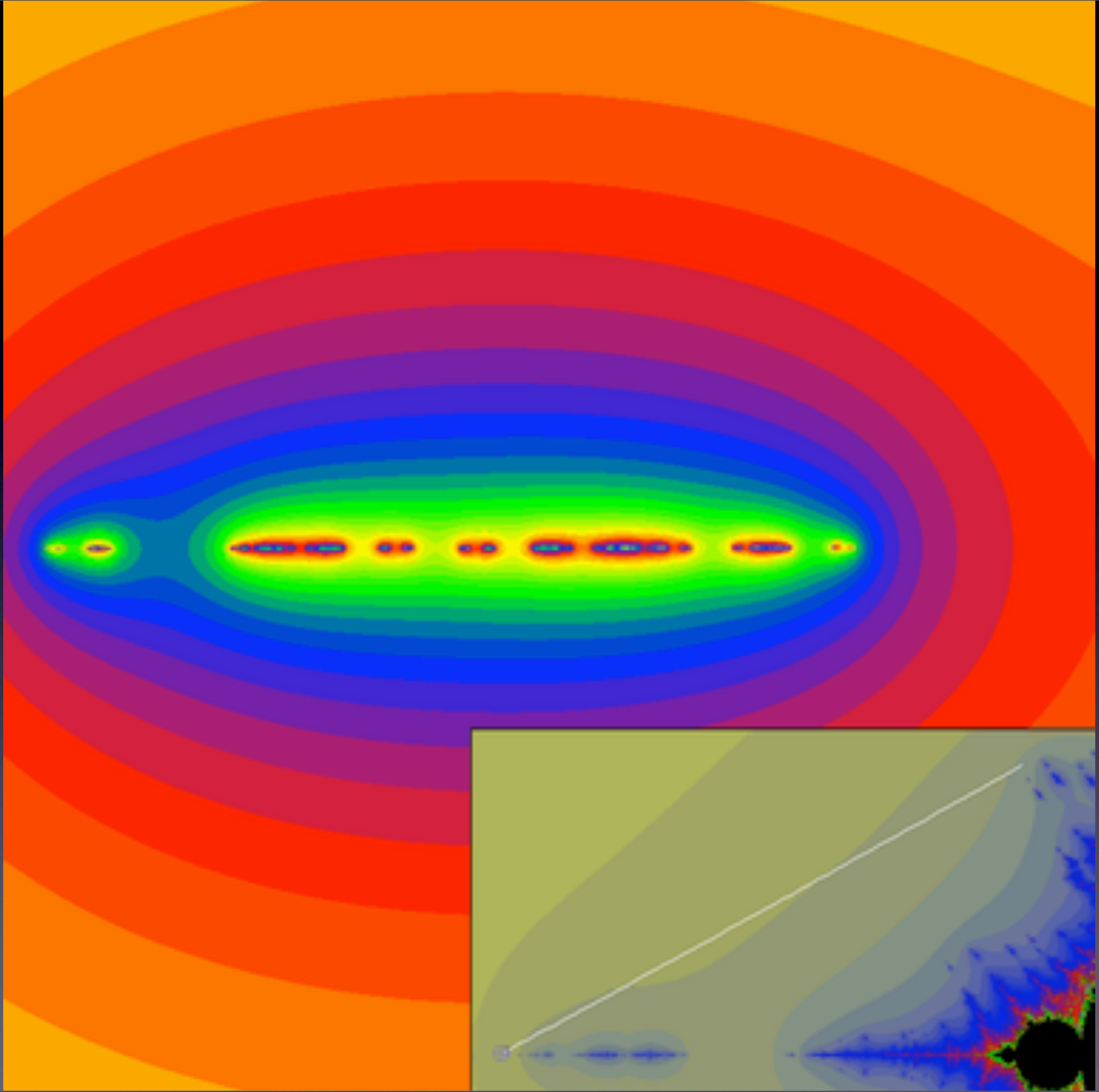
They are produced by the program
Saddledrop
written by Karl Papadantonakis

The hope is that the colored points are all horseshoes.
At the moment, our only way of proving such things
is to use Zin Arai's program, which is very time-consuming

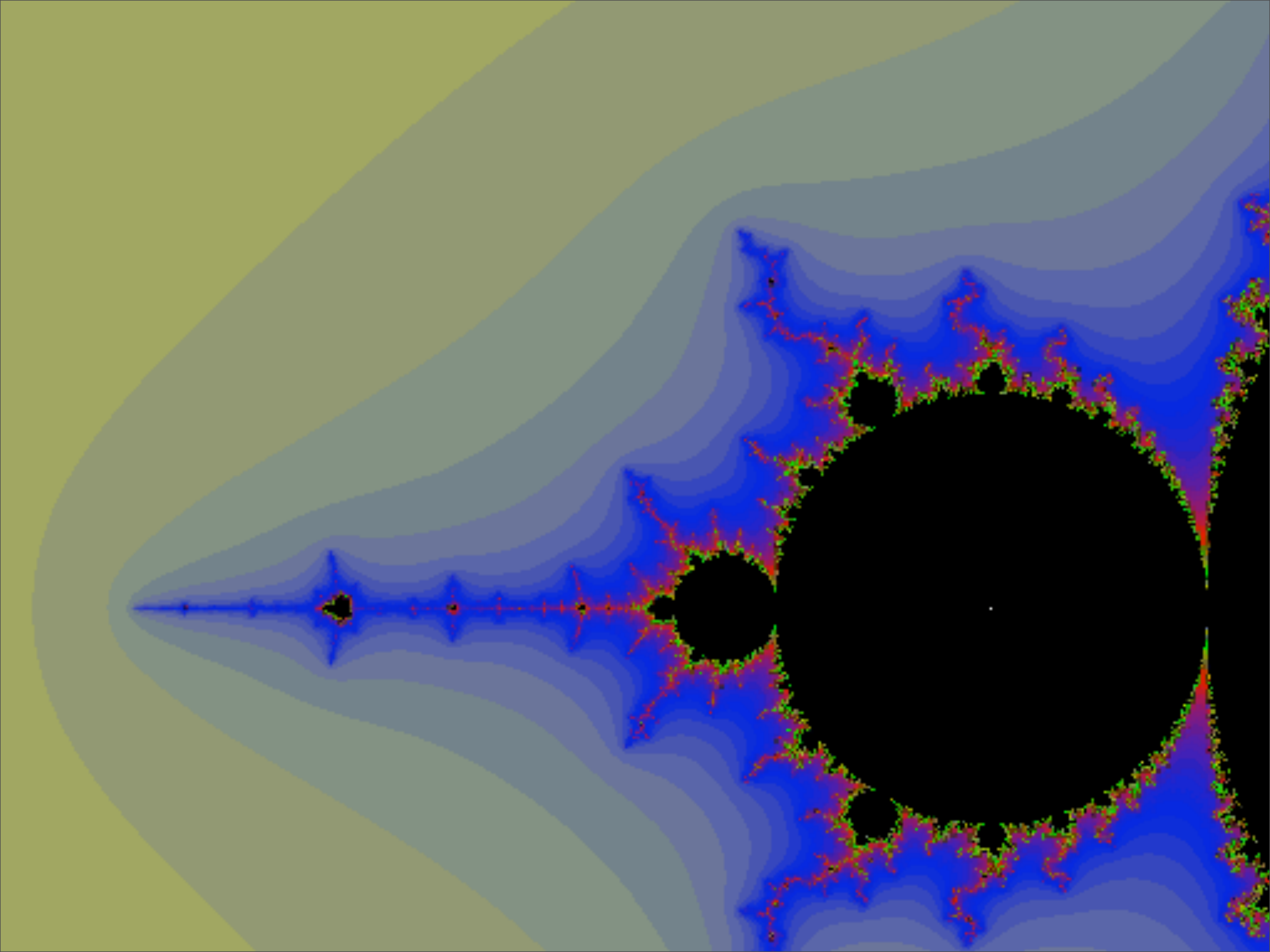

c-Plane Parameter Picture
a = -0.15 + 0.15i

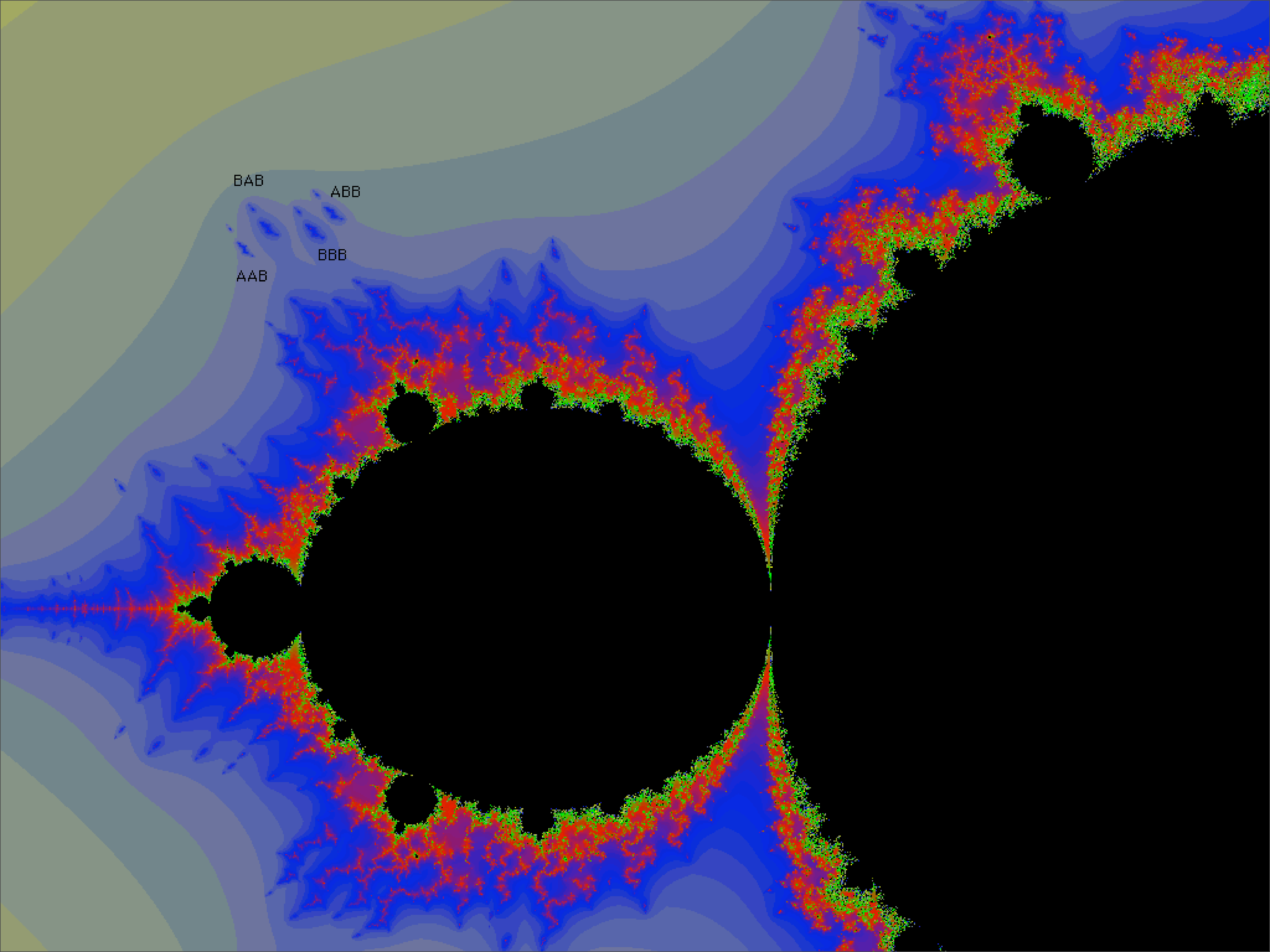

c-Plane Parameter Space Picture
a = 0.3

We will now show a movie of the monodromy of a particular loop in parameter space.



To understand the monodromy of this loop, we (i.e., Chris Lipa) sees where the reefs in parameter space come from in M.




BAB
ABB
BBB
AAB

What automorphism are we seeing?

What automorphism are we seeing?

- It is   AAB*BABAA

What automorphism are we seeing?

- It is AAB*BABAA
- This means that any time you see such a sequence, you exchange whatever symbol * is for the opposite symbol.

What automorphism are we seeing?

- It is   AAB*BABAA
- This means that any time you see such a sequence, you exchange whatever symbol * is for the opposite symbol.
- The formula is obtained from the kneading sequence *BABAA of all points of M beyond the *BABA polynomial.

What automorphism are we seeing?

- It is   AAB*BABAA
- This means that any time you see such a sequence, you exchange whatever symbol * is for the opposite symbol.
- The formula is obtained from the kneading sequence *BABAA of all points of M beyond the *BABA polynomial.
- AAB   the name of the sheet of cone over Cantor x Circle that the loop surrounds.

This suggests a great many questions:

What is the image of the monodromy homomorphism?

Conjecture. The image of the monodromy homomorphism, together with the shift, generate $\mathrm{Aut}(S_2)$.

Is it true that for every marker automorphism as above there actually in a loop in the horseshoe locus L inducing it?

Conjecture. Yes, and these automorphisms, together with the shift, generate $\mathrm{Aut}(S_2)$.

Pictures and movies made with PlanarIterations by Ben Hinkle and FractalAsm and SaddleDrop by Karl Papadantonakis, all available at www.math.cornell.edu/~dynamics/.

Pictures and movies made with PlanarIterations by Ben Hinkle and FractalAsm and SaddleDrop by Karl Papadantonakis, all available at www.math.cornell.edu/~dynamics/.

That's all folks