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A Fatou–Bieberbach domain avoiding a neighborhood of a variety of codimension 2

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1 Introduction

In response to a question of Y.-T. Siu, we show that for any algebraic variety V of codimension 2 in $\mathbb{C}^n$, there is a neighborhood U of V and an injective holomorphic map $\Phi : \mathbb{C}^n \to \mathbb{C}^n \setminus U$. That is, there is a Fatou-Bieberbach domain (a proper subdomain in $\mathbb{C}^n$ biholomorphic to $\mathbb{C}^n$) in the complement of some neighborhood of V. In particular, Φ is a dominating map. In case $n = 1$, V is empty, so the result is trivial, while if $n = 2$, V is a finite set, and it is well-known that there is a Fatou-Bieberbach domain omitting an open set, thus by scaling there is such a domain avoiding a neighborhood of V. Hence for the remainder of the paper we assume $n \geq 3$.

It should be noted that in general there is no corresponding result for nonalgebraic varieties: using techniques similar to those in [BF], Forstneric showed in [F] that there is a proper holomorphic embedding of $\mathbb{C}^{n-2}$ into $\mathbb{C}^n$ such that the image of any holomorphic map $\Phi : \mathbb{C}^2 \to \mathbb{C}^n$ with generic rank 2 must intersect the embedding of $\mathbb{C}^{n-2}$ infinitely often. This implies that there is no Fatou-Bieberbach domain in the complement of this embedding, let alone in the complement of a neighborhood.

There are two key ingredients in the present proof. The first is that after a suitable change of coordinates, the variety V is contained in some nice neighborhood of a linear subspace of dimension $n - 2$. The second is that there is a Fatou-Bieberbach domain which is contained in a relatively small neighborhood of a 1-dimensional subspace. A simple analysis of this situation allows us to translate these two neighborhoods to be disjoint, thus giving the result.

For notation, $z = (z_1, \ldots, z_n)$ is a point in $\mathbb{C}^n$, and $\|z\|_\infty$ is the max over all $|z_j|$.

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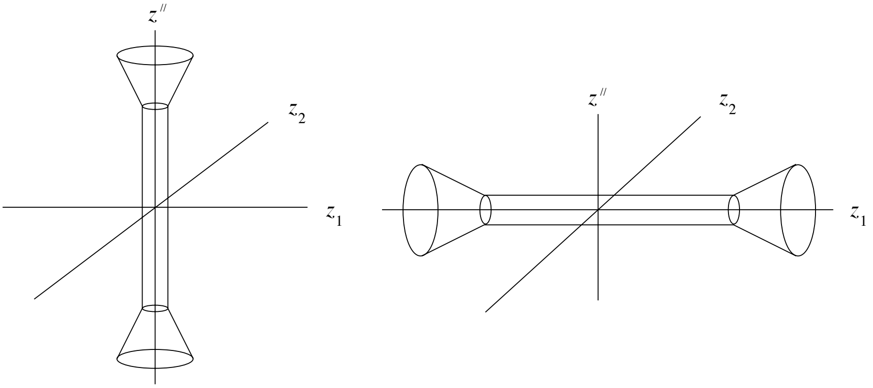

Fig. 1. On the left is the union of A_ϵ and B_ϵ; here the tube has radius ϵ. On the right is the union of D_1 and D_2; here the tube has radius R

2 Preparing the variety

In this section we choose coordinates to construct a nice neighborhood of a biholomorphic image of V. To state this more precisely, for $z \in \mathbb{C}^n$, let $z' = (z_1, z_2)$ and let $z'' = (z_3, \ldots, z_n)$. For $\epsilon > 0$, let

$$A_\epsilon = \{z \in \mathbb{C}^n : \|z'\|_\infty < \epsilon\},$$

and let

$$B_\epsilon = \{z \in \mathbb{C}^n : \|z'\|_\infty < \epsilon \|z''\|_\infty\}.$$

See Fig. 1 for a depiction of the union of these sets.

Lemma 2.1 *Let $V \subseteq \mathbb{C}^n$ be an algebraic variety of codimension 2. Then there exist coordinates $(z_1, \ldots, z_n)$ such that for all $\epsilon > 0$, there exists an invertible, complex linear map L_ϵ such that $L_\epsilon(V) \subseteq A_\epsilon \cup B_\epsilon$.*

Proof. Since V is algebraic, it extends to a variety $\overline{V}$ of codimension 2 in $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}_\infty$, which we may assume to have no component of dimension $n-2$ contained in $\mathbb{P}^{n-1}_\infty$. Let $V' = \overline{V} \cap \mathbb{P}^{n-1}_\infty$, in which case $\dim(V') = n-3$. A generic $\mathbb{P}^2$ in $\mathbb{P}^n$ intersects $\mathbb{P}^{n-1}_\infty$ in a $\mathbb{P}^1$, so we may choose some such $\mathbb{P}^2$, denoted $\mathbb{P}^2_0$, to avoid V'. Since $\mathbb{P}^2_0$ and $\overline{V}$ are closed, there exists a neighborhood U of $\mathbb{P}^2_0 \cap \mathbb{P}^{n-1}_\infty$ such that $\overline{U}$ and $\overline{V}$ are disjoint.

Changing coordinates, we may assume that

$$\mathbb{P}^2_0 \cap \mathbb{C}^n = \{(z_1, z_2, 0, \ldots, 0) : z_j \in \mathbb{C}\},$$

and shrinking U if necessary, we may assume that

$$U \cap \mathbb{C}^n = \{z : \|z'\|_\infty > r, \|z'\|_\infty > r\|z''\|_\infty\}$$

for some large r.

Let $L_\epsilon(z', z'') = (\epsilon z'/r, z'')$. Then $L_\epsilon(\mathbb{C}^n \setminus \overline{U}) = A_\epsilon \cup B_\epsilon$, and since V is contained in $\mathbb{C}^n \setminus \overline{U}$, the lemma follows. □

3 A thin Fatou-Bieberbach domain

In this section, we construct a Fatou-Bieberbach domain contained in a relatively small neighborhood of the z_1-axis. This result can be strengthened considerably in several ways, some of which are discussed after the proof of this lemma.

For notation, let $\Delta(\zeta; r)$ denote the circle in $\mathbb{C}$ with center ζ and radius r, and let $\Delta^k(0; r)$ be the k-fold product of $\Delta(0; r)$.

Lemma 3.1 *There exists a Fatou-Bieberbach domain D in $\mathbb{C}^n$ such that for some $R > 4$, D is contained in the union of $D_1 = \Delta(0; R^2) \times \Delta^{n-1}(0; R)$ and $D_2 = \{z : |z_1| \geq R^2 - 3R + \|(z_2, \ldots, z_n)\|_\infty\}$.*

Proof. Let

$$f(z) = (z_2, \ldots, z_n, (z_2^2 - z_1)/2),$$

and let

$$g(w) = f^{-1}(w) = (w_1^2 - 2w_n, w_1, \ldots, w_{n-1}).$$

The origin is a fixed point for f (and g). Calculating the derivative gives

$$(D_0 f)(v_1, \ldots, v_n)^T = (v_2, \ldots, v_n, -v_1/2)^T.$$

Hence an eigenvalue λ must satisfy $v_2 = \lambda v_1, \ldots, v_n = \lambda v_{n-1}$, and $-v_1/2 = \lambda v_n$. Hence $-v_1/2 = \lambda^n v_1$, so the eigenvalues are the nth roots of $-1/2$. In particular, the origin is an attracting fixed point for f, so by e.g. [RR], the basin of attraction $D = \{z : f^m(z) \to 0 \text{ as } m \to \infty\}$ is a Fatou-Bieberbach domain.

Since $g = f^{-1}$, it follows that for any $R > 0$, $D \subseteq \cup_{m \geq 0} g^m(\Delta^n(0; R))$. By [BP], if R is sufficiently large, then D is contained in $\Delta^n(0; R) \cup \{z : |z_1| \geq R, \|z\|_\infty = |z_1|\}$.

To prove the lemma, note that the form of g shows that

$$g(\Delta^n(0; R)) \subseteq \Delta(0; R^2 + 2R) \times \Delta^{n-1}(0; R).$$

Moreover, if w is contained in this latter set but not in $\Delta^n(0; R)$, then $z = g(w)$ satisfies $|z_1| - \| (z_2, \ldots, z_n)\|_\infty \geq |w_1|^2 - 2R - |w_1|$. Since $|w_1|(|w_1| - 1) \geq R(R-1)$, we have $z \in D_2$. Hence $g(D_1) \subseteq D_1 \cup D_2$. A similar argument shows that $g(D_2) \subseteq D_2$, so $g(D_1 \cup D_2) \subseteq D_1 \cup D_2$ and hence $D \subseteq D_1 \cup D_2$. □

Remark. In some sense, the Fatou-Bieberbach domain constructed above is contained in a relatively small neighborhood of the complex curve $\zeta \mapsto (\zeta^{2^{n-1}}, \zeta^{2^{n-2}}, \ldots, \zeta)$. To make this more precise, note that if w is contained in the set D_2 with $R > 4$, then

$$g^{\circ(n-1)}(w) = (w_1^{2^{n-1}} + O(|w_1|^{2^{n-1}-1}), w_1^{2^{n-2}} + O(|w_1|^{2^{n-2}-1}), \ldots, w_1),$$

where the constants implicit in O are independent of the $R > 4$ used to define D_2. In particular, for some $C > 0$, the Fatou-Bieberbach domain D is contained in the union of $g^{n-1}(D_1)$ and the set

$$\{(\zeta^{2^{n-1}}, \zeta^{2^{n-2}}, \ldots, \zeta) + \Delta(0; C|\zeta|^{2^{n-1}-1}) \\ \times \Delta(0; C|\zeta|^{2^{n-2}-1}) \times \cdots \times \Delta(0; C|\zeta|) \times \{0\} : |\zeta| > R\}.$$

4 Main Theorem

Theorem 4.1 *Let V be an algebraic variety of codimension 2 in $\mathbb{C}^n$. Then there exists a neighborhood U of V and an injective holomorphic map $\Phi : \mathbb{C}^n \to \mathbb{C}^n \setminus U$.*

Proof. Change coordinates as in lemma 2.1 and let U, L_ϵ, A_ϵ, and B_ϵ be as in that lemma. Let D and R be as in lemma 3.1.

Translating $D_1 \cup D_2$ by $(0, 2R, 0, \ldots, 0)$, the image is the union of

$$\hat{D}_1 = \Delta(0; R^2) \times \Delta(2R; R) \times \Delta^{n-1}(0; R)$$

and

$$\hat{D}_2 = \{z : |z_1| \geq R^2 - 3R + \|(z_2 - 2R, z_3, \ldots, z_n)\|_\infty\}.$$

For $\epsilon < 1$, if $z \in A_\epsilon \cup B_\epsilon$, then $\|(z_1, z_2)\|_\infty < 1$, so $z \notin \hat{D}_1 \cup \hat{D}_2$. Hence translating D by $(0, 2R, 0, \ldots, 0)$ and applying L_ϵ^{-1} gives a Fatou-Bieberbach domain in the complement of U, hence in the complement of a neighborhood of V, as desired. □

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